

# Local trace formulae for commuting Hamiltonians in Toeplitz quantization

Roberto Paoletti\*

## Abstract

Let  $(M, J, \omega)$  be a quantizable compact Kähler manifold, with quantizing Hermitian line bundle  $(A, h)$ , and associated Hardy space  $H(X)$ , where  $X$  is the unit circle bundle. Given a collection of  $r$  Poisson commuting quantizable Hamiltonian functions  $f_j$  on  $M$ , there is an induced Abelian unitary action on  $H(X)$ , generated by certain Toeplitz operators naturally induced by the  $f_j$ 's. As a multi-dimensional analogue of the usual Weyl law and trace formula, we consider the problem of describing the asymptotic clustering of the joint eigenvalues of these Toeplitz operators along a given ray, and locally on  $M$  the asymptotic concentration of the corresponding joint eigenfunctions. This problem naturally leads to a ‘directional local trace formula’, involving scaling asymptotics in the neighborhood of certain special loci in  $M$ . Under natural transversality assumption, we obtain asymptotic expansions related to the local geometry of the Hamiltonian action and flow.

## 1 Introduction

This paper is concerned with certain asymptotic expansions related to the singularities of a distributional trace, which is associated to the joint quantization of a family of pairwise commuting Hamiltonians in Toeplitz quantization. The emphasis will be on the local manifestation of these expansions, where locality is referred to the phase space of the the classical system, and to its relation to the underlying symplectic geometry and dynamics. Before stating the relevant results, we need to describe at some length the general picture in which we are working.

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\***Address:** Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano Bicocca, Via R. Cozzi 53, 20125 Milano, Italy; **e-mail:** roberto.paoletti@unimib.it

## 1.1 Quantization and distributional traces

### 1.1.1 Berezin-Toeplitz quantization in the Hardy space scheme

Let  $(M, J, \omega)$  be a  $d$ -dimensional compact Kähler manifold, endowed with the symplectic volume form  $dV_M =: \omega^{\wedge d}/d!$

The symplectic manifold  $(M, 2\omega)$  may be viewed as a model for a classical phase space. To any  $f \in \mathcal{C}^\infty(M)$ , the symplectic structure  $2\omega$  associates a Hamiltonian vector field  $v_f \in \mathfrak{X}(M)$ , and the latter generates a Hamiltonian flow  $\phi_s^M : M \rightarrow M$  ( $s \in \mathbb{R}$ ).

**Definition 1.1.** The Hamiltonian  $f \in \mathcal{C}^\infty(M)$  is *compatible* (with the Kähler structure  $(\omega, J)$  of  $M$ ) if  $\phi_\tau^M$  is holomorphic for every  $\tau \in \mathbb{R}$ .

Albeit very special, compatible Hamiltonians are a very important and natural object of study; for instance, they are closely related to holomorphic Lie group actions on complex projective manifolds.<sup>1</sup>

The following definition is standard in geometric quantization:

**Definition 1.2.** The Kähler manifold  $(M, J, \omega)$  is *quantizable* if there exists a positive Hermitian holomorphic line bundle  $(\mathcal{A}, h)$  on  $M$ , such that the unique compatible covariant derivative  $\nabla$  on  $\mathcal{A}$  has curvature  $\Theta = -2i\omega$ .

It is well-known that the spaces  $H^0(M, \mathcal{A}^{\otimes k})$  of global holomorphic sections of powers of  $\mathcal{A}$ , endowed with their natural Hilbert space structures, play the role of a ‘quantum counterpart’ of  $(M, 2\omega)$  at Plack’s constant  $\hbar = 1/k$ ,  $k = 1, 2, \dots$ . Hardy space formalism provides a convenient repackaging of this picture, in which sections can be viewed as functions, and all values of  $\hbar$  can be treated collectively, as we now recall.

Let  $\mathcal{A}^\vee$  be the dual line bundle to  $\mathcal{A}$ , with the induced Hermitian metric, and let  $X \subseteq \mathcal{A}^\vee$  be the unit circle bundle, with projection  $\pi : X \rightarrow M$  and connection 1-form  $\alpha \in \Omega^1(X)$ . Then  $(X, \alpha)$  is a contact manifold, with volume form  $dV_X =: (\alpha/2\pi) \wedge \pi^*(dV_M)$ . We shall denote by  $\partial_\theta$  the generator of the structure  $S^1$ -action on  $X$ .

The tangent bundle of  $X$  splits as an invariant direct sum

$$TX = \mathcal{V} \oplus \mathcal{H}, \tag{1}$$

where  $\mathcal{V} = \ker(d\pi)$ , the vertical tangent bundle, is the rank-1 sub-bundle generated by  $\partial_\theta$ , and  $\mathcal{H} = \ker(\alpha)$  is the horizontal tangent bundle. The complex structure  $J$  naturally lifts to a complex structure  $J_H$  on the horizontal tangent bundle of  $X$ , and  $J_H$  is a CR structure on  $X$ .

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<sup>1</sup>Compatible Hamiltonians are also called *quantizable* in the literature [CGR].

For any real-valued  $f \in \mathcal{C}^\infty(M)$ , there is a natural lift of  $v_f$  to a contact vector field  $\tilde{v}_f \in \mathfrak{X}(X)$ , given by

$$\tilde{v}_f =: v_f^\sharp - f \partial_\theta, \quad (2)$$

where  $v_f^\sharp$  is the  $\alpha$ -horizontal lift of  $v_f$ <sup>2</sup>. Thus  $\tilde{v}_f$  generates a contact flow  $\phi_s^X : X \rightarrow X$  on  $(X, \alpha)$ . This flow preserves the splitting (1); in addition, it preserves the CR structure  $J_H$  precisely when  $f$  is compatible.

The following is standard terminology (see [BtSj], [BtG], [Z] and [BSZ]):

**Definition 1.3.** The *Hardy space*  $H(X) \subseteq L^2(X)$  of  $X$  consists of the boundary values of  $L^2$ -summable holomorphic functions on the unit disc bundle of  $A^\vee$ . The *Szegő projector* of  $X$  is the  $L^2$ -orthogonal projector  $\Pi : L^2(X) \rightarrow H(X)$ ; its distributional kernel  $\Pi \in \mathcal{D}'(X \times X)$  is called the *Szegő kernel* of  $X$ .

As is well-known, there is a natural unitary isomorphism

$$H(X) \cong H(M, \mathcal{A}) =: \bigoplus_{\ell \geq 0} H^0(M, \mathcal{A}^{\otimes \ell}),$$

where  $\bigoplus$  is the Hilbert space direct sum. The subspace of  $H(X)$  corresponding to  $H^0(M, \mathcal{A}^{\otimes \ell})$  is precisely the  $\ell$ -th equivariant piece  $H(X)_\ell \subseteq H(X)$  for the structure  $S^1$ -action [Z], [BSZ].

By pull-back, the flow  $\phi_s^X$  yields a 1-parameter family of unitary automorphisms  $U(s) = U_f(s) =: (\phi_s^X)^* : L^2(X) \rightarrow L^2(X)$  ( $s \in \mathbb{R}$ ). Furthermore,  $f$  is compatible if and only if  $\phi_s^X$  preserves the CR structure of  $X$ , and this is equivalent to  $H(X)$  being  $U(s)$ -invariant for every  $s$ . Therefore, a compatible  $f$  induces a 1-parameter family of unitary automorphisms

$$\mathfrak{U}(s) = \mathfrak{U}_f(s) : H(X) \rightarrow H(X). \quad (3)$$

The family  $\mathfrak{U}(s)$  is a quantization of the Hamiltonian flow  $\phi_s^M$ ; the quantization of the classical Hamiltonian  $f$  should be a self-adjoint operator acting on  $H(X)$ , and in Berezin-Toeplitz quantization this is given by a Toeplitz operator. In the Hardy space picture, following [BtG], these are defined as follows.

**Definition 1.4.** A  $k$ -th order Toeplitz operator on  $X$  is a composition  $T =: \Pi \circ Q \circ \Pi$ , where  $Q$  is a  $k$ -th order pseudo-differential operator;  $T$  is viewed as a possibly unbounded linear operator on  $H(X)$ .

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<sup>2</sup> Modifying  $f$  by an additive constant will leave  $v_f$  unchanged, but alter  $\tilde{v}_f$ .

By the theory of [BtG], Toeplitz operators have a well-defined principal symbol. Let

$$\Sigma =: \{(x, r\alpha_x) : x \in X, r > 0\} \subseteq T^*X \setminus (0).$$

be the closed symplectic cone sprayed by  $\alpha$ .

**Definition 1.5.** If  $T$  is a  $k$ -th order Toeplitz operator on  $X$ , its principal symbol  $\mathfrak{s}_T : \Sigma \rightarrow \mathbb{C}$  is the  $k$ -th order homogeneous function on  $\Sigma$  given by the restriction of the principal symbol of  $Q$  ( $\mathfrak{s}_T$  is independent of the choice of  $Q$  in the definition of  $T$ ).

For instance, given  $f \in \mathcal{C}^\infty(M)$  real valued, let  $M_f : L^2(X) \rightarrow L^2(X)$  be the self-adjoint operator given by multiplication by  $f \circ \pi$ . Then  $T_f =: \Pi \circ M_f \circ \Pi$  is an invariant zeroth order Toeplitz operator, viewed as a self-adjoint endomorphism of  $H(X)$ . Its principal symbol is  $\mathfrak{s}_{T_f}(x, r\alpha_x) =: f(\pi(x))$ . Composing  $T_f$  with the ‘number operator’  $D = -i\partial_\theta$  turns it into a first order operator  $T'_f$ , with principal symbol  $\mathfrak{s}_{T'_f}(x, r\alpha_x) = r f(\pi(x))$ .

When  $f$  is compatible, there is another first-order Toeplitz operator associated to it, that captures more explicitly the associated dynamics. Namely,  $\tilde{v}_f$  is a skew-Hermitian operator on  $L^2(X)$  and leaves  $H(X)$  invariant; therefore, the restriction

$$\mathfrak{T}_f =: i\tilde{v}_f|_{H(X)} : H(X) \rightarrow H(X) \tag{4}$$

is a first-order (formally) self-adjoint Toeplitz operator; its principal symbol is again  $\mathfrak{s}_{\mathfrak{T}_f}(x, r\alpha_x) = r f(\pi(x))$ . Then  $\mathfrak{T}_f$  generates  $\mathfrak{U}(\cdot)$ , i.e.  $\mathfrak{U}(s) = e^{is\mathfrak{T}_f}$ .

### 1.1.2 Distributional traces of Toeplitz operators

Geometric quantization aims to relate the asymptotic properties of a quantized system to the underlying classical dynamics and geometry. These properties may be of either global or local nature on  $(M, 2\omega)$ . For instance, the spectral asymptotics of a Toeplitz operator yield information of a global nature, while the asymptotic concentration of its eigenfunctions is a local result. Local properties can be turned into global ones by integration.

In particular, suppose that  $f > 0$ . Then  $\mathfrak{T}_f$  has eigenvalues on  $H(X)$

$$\lambda_1 \leq \lambda_2 \leq \dots,$$

repeated according to multiplicity, with  $\lambda_j \uparrow +\infty$ ; there is a complete orthonormal system  $(e_j)$  of  $H(X)$  formed by corresponding eigenvectors.

The distributional kernels of  $\mathfrak{T}_f$  and  $\mathfrak{U}(s)$  may be represented in terms of these spectral data<sup>3</sup>:

$$\mathfrak{T}_f(x, y) = \sum_j \lambda_j e_j(x) \cdot \overline{e_j(y)}, \quad \mathfrak{U}(s, x, y) = \sum_j e^{is\lambda_j} e_j(x) \cdot \overline{e_j(y)}, \quad (5)$$

where  $x, y \in X$  and  $s \in \mathbb{R}$ .

The distributional trace

$$\mathrm{tr}(\mathfrak{U}) =: \sum_j e^{i\lambda_j s} : \chi = \chi(s) \mapsto \sum_j \widehat{\chi}(-\lambda_j)$$

is then a well-defined distribution on the real line, and its singularities are concentrated on the set of periods of the contact flow  $\phi^X$  [BtG]. The trace formula in *loc. cit.* describes the singularity at each period (for Toeplitz operators related to general symplectic cones). In particular, by a Tauberian argument the estimate of the ‘big’ singularity at the origin yields a Weyl law for the counting function of the  $\lambda_j$ ’s<sup>4</sup>.

In the present Berezin-Toeplitz context, local version of these results (i.e., local Weyl laws and local trace formulae) were obtained in [P2], [P3], [P5], [P6].

### 1.1.3 Commuting Hamiltonians and Abelian contact actions

We aim to generalize these results to a collection of Poisson commuting compatible Hamiltonians  $f_1, \dots, f_r \in \mathcal{C}^\infty(M)$ , meaning that  $\{f_k, f_l\} = 0$  for  $k, l = 1, \dots, r$ , where  $\{, \}$  is the usual Poisson Lie bracket of  $\mathcal{C}^\infty(M)$ . Let us write  $v_k$  for  $v_{f_k}$ , and similarly for  $\tilde{v}_k$ . Then  $[v_k, v_l] = 0$  on  $M$ .

In addition, under the previous hypothesis, for every  $j, k = 1, \dots, r$ , we have on  $X$

$$[\tilde{v}_j, \tilde{v}_k] = [v_j, v_k]^\# - \{f_j, f_k\} \partial_\theta = 0.$$

Therefore, we obtain commuting self-adjoint first order Toeplitz operators  $\mathfrak{T}_k$ , given by the restriction to  $H(X)$  of  $i\tilde{v}_k$ ,  $k = 1, \dots, r$ .

For instance, suppose that an  $\ell$ -dimensional compact torus  $\mathbf{T}$  acts on  $M$  in a holomorphic and Hamiltonian manner, and let  $\Phi : M \rightarrow \mathrm{Lie}(\mathbf{T})^\vee$  be the moment map to the Lie coalgebra of  $\mathbf{T}$ . If  $\mathbf{v}_j \in \mathrm{Lie}(\mathbf{T})$ ,  $j = 1, \dots, r$ , then the functions  $f_j =: \langle \Phi, \mathbf{v}_j \rangle$  are compatible and Poisson commute.

Let  $\phi_{j,s}^M : M \rightarrow M$  and  $\phi_{j,s}^X : X \rightarrow X$  be the Hamiltonian and contact flows associated to each  $f_j$  ( $s \in \mathbb{R}$ ). Thus

$$\phi_{k,s}^M \circ \phi_{l,s'}^M = \phi_{l,s'}^M \circ \phi_{k,s}^M \quad \text{and} \quad \phi_{k,s}^X \circ \phi_{l,s'}^X = \phi_{l,s'}^X \circ \phi_{k,s}^X, \quad (6)$$

<sup>3</sup>We shall not distinguish notationally an operator from its distributional kernel.

<sup>4</sup>For pseudodifferential operators, corresponding results had appeared in [H1] and [DG]

for all  $k, l = 1, \dots, r$  and  $s, s' \in \mathbb{R}$ .

Let us define  $\phi^M : \mathbb{R}^r \times M \rightarrow M$  by

$$\phi^M(\mathbf{s}, \cdot) = \phi_{\mathbf{s}}^M =: \phi_{1,s_1}^M \circ \dots \circ \phi_{r,s_r}^M : M \rightarrow M \quad (\mathbf{s} = (s_j) \in \mathbb{R}^r);$$

in view of (6), this is an holomorphic action. It is furthermore Hamiltonian, with moment map

$$\Phi = (f_1, \dots, f_r)^t : M \rightarrow (\mathbb{R}^r)^\vee \cong \mathbb{R}^r, \quad (7)$$

where the latter isomorphism is by means of the standard scalar product.

In the same manner, we obtain a contact action  $\phi^X : \mathbb{R}^r \times X \rightarrow X$ , given by

$$\phi^X(\mathbf{s}, \cdot) = \phi_{\mathbf{s}}^X =: \phi_{1,s_1}^X \circ \dots \circ \phi_{r,s_r}^X : X \rightarrow X \quad (\mathbf{s} = (s_j) \in \mathbb{R}^r),$$

which lifts  $\phi^M$  in a natural manner. Pulling-back, we have the unitary representations of  $\mathbb{R}$

$$\mathfrak{U}_j(s) =: (\phi_{j,-s}^X)^* : H(X) \rightarrow H(X),$$

which may be combined into a unitary representation  $\mathfrak{U} : \mathbb{R}^r \times H(X) \rightarrow H(X)$ , given by

$$\mathfrak{U}(\mathbf{s}) = \mathfrak{U}(\mathbf{s}, \cdot) =: \mathfrak{U}_1(s_1) \circ \dots \circ \mathfrak{U}_r(s_r) = (\phi_{-\mathbf{s}}^X)^* : H(X) \rightarrow H(X). \quad (8)$$

#### 1.1.4 The joint spectrum and the associated trace

Each  $\mathfrak{T}_k$  is  $S^1$ -invariant, and therefore preserves the finite-dimensional  $S^1$ -equivariant pieces  $H(X)_\ell$ ,  $\ell = 0, 1, 2, \dots$ ; the spectrum of  $\mathfrak{T}_k$  is the union over  $\ell \in \mathbb{N}$  of the finite spectra of its restrictions to the  $H(X)_\ell$ 's.

Furthermore, there is a complete orthonormal system  $(e_j)$  of  $H(X)$  composed of joint eigenvectors of the  $\mathfrak{T}_k$ 's. That is, for each  $j = 1, 2, \dots$  and  $k = 1, \dots, r$  we have

$$\mathfrak{T}_k(e_j) = \lambda_{kj} e_j,$$

where  $\Lambda_j =: (\lambda_{1j}, \dots, \lambda_{rj})^t \in \mathbb{R}^r$  is a *joint eigenvalue* of the  $\mathfrak{T}_k$ 's.

We see in particular that for every  $j$  we have

$$\begin{aligned} \mathfrak{U}(\mathbf{s})(e_j) &= \mathfrak{U}_1(s_1) \circ \dots \circ \mathfrak{U}_r(s_r)(e_j) \\ &= e^{i(\lambda_{1j}s_1 + \dots + \lambda_{rj}s_r)} e_j = e^{i\langle \Lambda_j, \mathbf{s} \rangle} e_j. \end{aligned} \quad (9)$$

In general, a given joint eigenvalue  $\beta \in \mathbb{R}^r$  of the commuting system  $\mathfrak{T} =: (\mathfrak{T}_k)$  needn't have finite multiplicity: it may happen that  $\beta = \Lambda_j$  for

infinitely many  $j$ 's. Nonetheless, as in the case  $r = 1$ , infinite multiplicities do not occur if  $\mathbf{0} \notin \Phi(M)$  because in this case  $\Lambda_j \rightarrow \infty$  (Lemma 2.1).

If  $\mathbf{0} \notin \Phi(M)$ , therefore, the  $\Lambda_j$ 's drift to infinity and (just to fix ideas) may be ordered lexicographically in a non-decreasing sequence  $\Lambda_1 \leq \Lambda_2 \leq \dots$ , where each joint eigenvalue appears repeated according to its multiplicity. For each  $\mathbf{s} = (s_j) \in \mathbb{R}^r$ , we obtain a first order self-adjoint Toeplitz operator of the form

$$\langle \mathfrak{T}, \mathbf{s} \rangle =: \sum_{k=1}^r s_k \mathfrak{T}_k,$$

with eigenvalues  $\langle \Lambda_j, \mathbf{s} \rangle = \sum_{k=1}^r \lambda_{kj} s_k$  relative to the eigenvectors  $e_j$ . Clearly,  $\langle \mathfrak{T}, \mathbf{s} \rangle$  is the restriction to  $H(X)$  of  $i \tilde{v}_{\Phi \mathbf{s}}$ , where

$$\tilde{v}_{\Phi \mathbf{s}} = v_{\Phi \mathbf{s}}^\# - \Phi^\mathbf{s} \partial_\theta, \quad (10)$$

and  $\Phi^\mathbf{s} =: \langle \Phi, \mathbf{s} \rangle = \sum_{k=1}^r s_k f_k$ , and its Schwartz kernel is

$$\langle \mathfrak{T}, \mathbf{s} \rangle(x, y) = \sum_{j=1}^{+\infty} \langle \Lambda_j, \mathbf{s} \rangle e_j(x) \cdot \overline{e_j(y)} \quad (x, y \in X, \mathbf{s} \in \mathbb{R}^r).$$

Similarly, we see from (9) that

$$\mathfrak{U}(\mathbf{s}, x, y) = \sum_{j=1}^{+\infty} e^{i \langle \Lambda_j, \mathbf{s} \rangle} e_j(x) \cdot \overline{e_j(y)} = e^{i \langle \mathfrak{T}, \mathbf{s} \rangle}(x, y). \quad (11)$$

Then the distributional trace

$$\mathrm{tr}(\mathfrak{U}) =: \sum_j e^{i \langle \Lambda_j, \cdot \rangle} \quad (12)$$

is a well-defined temperate distribution on  $\mathbb{R}^r$ , whose singularities encapsulate asymptotic information on the distribution of the  $\Lambda_j$ 's.

As in the 1-dimensional case, the singular support of  $\mathrm{tr}(\mathfrak{U})$  is contained in the set of periods of  $\phi^X$ ,

$$\mathrm{Per}(\phi^X) =: \{ \mathbf{s} \in \mathbb{R}^r : \exists x \in X \text{ such that } \phi_\mathbf{s}^X(x) = x \}; \quad (13)$$

however, unlike the case  $r = 1$ ,  $\mathrm{Per}(\phi^X)$  needn't consist of isolated points for  $r \geq 2$ .

So let us fix a period  $\mathbf{s}_0 \in \mathrm{Per}(\phi^X)$  and a covector  $\beta \in (\mathbb{R}^r)^\vee$  of unit length. As a measure of the singularity of  $\mathrm{tr}(\mathfrak{U})$  at  $\mathbf{s}_0$  in the direction  $\beta$ , we can consider the asymptotics for  $\lambda \rightarrow \infty$  of the Fourier transform

$$\mathcal{F}(\chi_{\mathbf{s}_0} \cdot \mathrm{tr}(\mathfrak{U}))(\lambda \beta) = \langle \mathrm{tr}(\mathfrak{U}), \chi_{\mathbf{s}_0} e^{-i\lambda \langle \beta, \cdot \rangle} \rangle, \quad (14)$$

where  $\chi_{\mathbf{s}_0}$  is a bump function supported in a small neighborhood of  $\mathbf{s}_0$ . We shall take  $\chi_{\mathbf{s}_0}(\cdot) =: \chi(\cdot - \mathbf{s}_0)$ , where  $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^r)$  is a bump function vanishing for  $\|\mathbf{s}\| \geq \epsilon$ . We then obtain for (14):

$$\begin{aligned} \mathcal{F}(\chi_{\mathbf{s}_0} \cdot \text{tr}(\mathfrak{U}))(\lambda \beta) &= \sum_j \langle e^{i\langle \Lambda_j, \cdot \rangle}, \chi_{\mathbf{s}_0} e^{-i\lambda \langle \beta, \cdot \rangle} \rangle \\ &= e^{-i\lambda \langle \beta, \mathbf{s}_0 \rangle} \sum_j e^{i\langle \Lambda_j, \mathbf{s}_0 \rangle} \widehat{\chi}(\lambda \beta - \Lambda_j). \end{aligned} \quad (15)$$

In particular, for  $\mathbf{s}_0 = \mathbf{0}$  (15) reduces to

$$\mathcal{F}(\chi \cdot \text{tr}(\mathfrak{U}))(\lambda \beta) = \sum_j \widehat{\chi}(\lambda \beta - \Lambda_j), \quad (16)$$

which, for  $\lambda \rightarrow +\infty$ , detects the rate at which the  $\Lambda_j$ 's asymptotically accumulate in the neighborhood of the ray  $\mathbb{R}_+ \beta$ .

On the other hand, (15) may be expressed as the genuine trace of the smoothing operator

$$\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0) =: \int_{\mathbb{R}^r} \chi_{\mathbf{s}_0}(\mathbf{s}) e^{-i\lambda \langle \beta, \mathbf{s} \rangle} \mathfrak{U}(\mathbf{s}) \, d\mathbf{s}. \quad (17)$$

In other words, if  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, \cdot, \cdot) \in \mathcal{C}^\infty(X \times X)$  denotes the Schwartz kernel of  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0)$ , then

$$\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, x, y) = \sum_j e^{-i\langle \lambda \beta - \Lambda_j, \mathbf{s}_0 \rangle} \widehat{\chi}(\lambda \beta - \Lambda_j) e_j(x) \cdot \overline{e_j(y)}. \quad (18)$$

and

$$\mathcal{F}(\chi_{\mathbf{s}_0} \cdot \text{tr}(\mathfrak{U}))(\lambda \beta) = \int_X \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, x, x) \, dV_X(x). \quad (19)$$

It is suggestive to view  $\mathcal{S}_\chi(\lambda \beta, \mathbf{0})$  as a ‘smoothed spectral projector’, corresponding to a cluster of joint eigenvalues traveling to infinity along the ray  $\mathbb{R}_+ \beta$ .

Here we shall analyze the local asymptotics of  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, \cdot, \cdot)$ . Although our methods apply with minor changes to the general case, to simplify the exposition we shall restrict our treatment to the on-diagonal asymptotics (which is the one relevant to trace applications). For instance, for  $\mathbf{s} = \mathbf{0}$  we obtain

$$\mathcal{S}_\chi(\lambda \beta, \mathbf{0}, x, x) = \sum_j \widehat{\chi}(\lambda \beta - \Lambda_j) |e_j(x)|^2, \quad (20)$$

which detects the asymptotic distribution of the ‘probability amplitudes’ of the eigenfunctions corresponding to joint eigenvalues asymptotically clustering along the axis  $\mathbb{R}_+ \beta$ .



### 1.1.5 Moment map directional transversality

Before stating our results, we need to introduce some further pieces of notation and definitions.

**Notation 1.1.** We shall view  $\mathbb{R}^r$  as an Abelian Lie group, with Lie algebra  $T_0\mathbb{R}^r \cong \mathbb{R}^r$  itself, and coalgebra  $(\mathbb{R}^r)^\vee$ , which we shall identify with  $\mathbb{R}^r$  by means of the standard scalar product. Since it will be convenient to distinguish the various roles of  $\mathbb{R}^r$  in our arguments, we shall write  $\mathfrak{t} =: T_0\mathbb{R}^r$ , and write the moment map (7) as  $\Phi = (f_k) : M \rightarrow \mathfrak{t}^\vee$ . We shall generally denote elements of  $\mathbb{R}^r$ , viewed as group elements, by  $\mathbf{s}_0, \mathbf{s}, \dots$ , elements of  $\mathfrak{t}$ , viewed as tangent vectors at the origin, by  $\xi, \eta, \dots$ , and the general element of  $\mathfrak{t}^\vee$  as  $\beta$ .

**Definition 1.6.** Any  $\xi \in \mathfrak{t}$  induces in a standard manner vector fields

$$\xi_M \in \mathfrak{X}(M) \text{ and } \xi_X \in \mathfrak{X}(X)$$

on  $M$  and  $X$ , respectively. For any  $m \in M$  and  $x \in X$ , we then have evaluation maps  $\text{val}_m : \mathfrak{t} \rightarrow T_m M$  and  $\text{val}_x : \mathfrak{t} \rightarrow T_x X$ , given by

$$\text{val}_m : \xi \mapsto \xi_M(m) \text{ and } \text{val}_x : \xi \mapsto \xi_X(x),$$

respectively.

*Remark 1.1.* Let  $(e_1, \dots, e_r)$  be the canonical basis of  $\mathfrak{t} = \mathbb{R}^r$ . In intrinsic notation,  $\Phi = \sum_j f_j e_j^*$ , where  $(e_j^*)$  is the dual basis. We have, in particular,  $e_{jM} = v_j$ ,  $\Phi^{e_j} = \langle \Phi, e_j \rangle = f_j$ ,  $e_{jX} = \tilde{v}_j$ . More generally, for any  $\xi \in \mathfrak{t}$ ,  $\xi_M$  is the Hamiltonian vector field associated to  $\Phi^\xi =: \langle \Phi, \xi \rangle$ , and  $\xi_X$  its contact lift according to (2):

$$\xi_X = \xi_M^\# - \Phi^\xi \partial_\theta.$$

**Definition 1.7.** Suppose  $\beta \in \mathfrak{t}^\vee$ ,  $\beta \neq 0$ . We shall set  $M_\beta =: \Phi^{-1}(\mathbb{R}_+ \beta)$  and  $X_\beta =: \pi^{-1}(M_\beta)$ .

Our local analysis requires that  $\Phi : M \rightarrow (\mathbb{R}^r)^\vee \cong \mathbb{R}^r$  be transverse to  $\mathbb{R}_+ \beta$ . Thus  $M_\beta$  is an invariant compact submanifold of  $M$  of codimension  $r - 1$ .

*Remark 1.2.* When  $\phi^M$  descends to an action of the torus  $\mathbf{T}^r = \mathbb{R}^r / \mathbb{Z}^r$ ,  $M_\beta$  is also connected (§2.1 of [P4]).

*Remark 1.3.* This transversality assumption is equivalent to the contact action  $\phi^X : \mathbb{R}^r \times X \rightarrow X$  being locally free on  $X_\beta$ . In turn, this is also equivalent to the following condition: for any  $m \in M_\beta$ , the restriction of  $\text{val}_m$  to  $\ker(\Phi(m)) \subseteq \mathfrak{t}$  is injective (§2.2 of [P4]; see §2.1.3 below).

**Definition 1.8.** Assume that  $\Phi : M \rightarrow \mathfrak{t}^\vee$  is transverse to  $\mathbb{R}_+ \beta$ , for some  $\beta \in \mathfrak{t}^\vee$  of unit norm. Then, in view of Remark 1.3, for any  $m \in M_\beta$  the vector subspace  $\ker \Phi(m) \subseteq \mathfrak{t}$  inherits two Euclidean structures

$$\langle \cdot, \cdot \rangle_0, \langle \cdot, \cdot \rangle_1 : \ker \Phi(m) \times \ker \Phi(m) \rightarrow \mathbb{R},$$

where the former is the restriction of the Euclidean product of  $\mathfrak{t}$ , and the latter is the pull-back of the Euclidean product on  $T_m M$  under  $\text{val}_m$ . Let  $\mathcal{K} = (v_l)$  be any *orthonormal* basis of  $\ker \Phi(m)$  with respect to  $\langle \cdot, \cdot \rangle_0$ , and let  $D(m) = D(m, \mathcal{K})$  be the representative matrix of  $\langle \cdot, \cdot \rangle_1$  with respect to  $\mathcal{K}$ , i.e.

$$D(m)_{kl} = \langle v_k, v_l \rangle_1 = g_m(v_{kM}(m), v_{lM}(m)),$$

where  $g$  is the Riemannian metric on  $M$ . Then  $\det D(m) > 0$  is independent of the choice of  $\mathcal{K}$ , and we can define a  $\mathcal{C}^\infty$  function  $\mathcal{D} : M_\beta \rightarrow \mathbb{R}_+$  by setting

$$\mathcal{D}(m) =: \sqrt{\det D(m)}.$$

### 1.1.6 Periods and singularities

Let us adopt the short-hand  $m_{\mathbf{s}} =: \phi_{-\mathbf{s}}^M(m)$  and  $x_{\mathbf{s}} =: \phi_{-\mathbf{s}}^X(x)$  ( $m \in M$ ,  $x \in X$ ,  $\mathbf{s} \in \mathbb{R}^r$ ).

**Definition 1.9.** For any  $\mathbf{s} \in \mathbb{R}^r$ , let us denote by

$$M(\mathbf{s}) =: \text{Fix}(\phi_{\mathbf{s}}^M) = \{m \in M : m = m_{\mathbf{s}}\}$$

and

$$X(\mathbf{s}) =: \text{Fix}(\phi_{\mathbf{s}}^X) = \{x \in X : x = x_{\mathbf{s}}\}$$

the fixed loci of  $\phi_{\mathbf{s}}^M : M \rightarrow M$  and  $\phi_{\mathbf{s}}^X : X \rightarrow X$ , respectively.

*Remark 1.4.* In general  $X(\mathbf{s})$  is the inverse image in  $X$  of the union of some connected components of  $M(\mathbf{s})$  (but perhaps not all of them).

**Definition 1.10.** The period sets of  $\phi^M$  and  $\phi^X$  are, respectively,

$$\text{Per}(\phi^M) =: \{\mathbf{s} \in \mathbb{R}^r : M(\mathbf{s}) \neq \emptyset\}$$

and

$$\text{Per}(\phi^X) =: \{\mathbf{s} \in \mathbb{R}^r : X(\mathbf{s}) \neq \emptyset\}.$$

If  $\mathbf{s} \in \text{Per}(\phi^X)$ , we shall set

$$X_\beta(\mathbf{s}) =: X_\beta \cap X(\mathbf{s}).$$

Clearly,  $\text{Per}(\phi^X) \subseteq \text{Per}(\phi^M)$ , and the inclusion is generally strict. We then have (see §2.1.1 and §2.2.2 below):

**Proposition 1.1.** *If  $x \in X$ , let us set  $m_x =: \pi(x)$ . Then the wave front set of  $\text{tr}(\mathfrak{U}) \in \mathcal{D}'(\mathbb{R}^r)$  is*

$$\text{WF}(\text{tr}(\mathfrak{U})) = \{(\mathbf{s}, r \Phi(m_x)) : \mathbf{s} \in \text{Per}(\phi^X), x \in X(\mathbf{s}), r > 0\}.$$

**Corollary 1.1.** *The singular support of  $\text{tr}(\mathfrak{U})$  is*

$$\text{SS}(\text{tr}(\mathfrak{U})) = \text{Per}(\phi^X).$$

### 1.1.7 Heisenberg local coordinates

Finally, our local scaling asymptotics are expressed in terms of a system  $\gamma_x$  of Heisenberg local coordinates (HLCS) on  $X$  centered at  $x \in X$ . We shall refer to [SZ] for a precise definition and a complete discussion of Heisenberg local coordinates (HLC), and simply list some of their salient properties.

A HLCS centered at  $x \in X$  is commonly represented in additive notation,

$$\gamma_x : (-\pi, \pi) \times B_{2d}(\mathbf{0}, \delta) \rightarrow X, \quad (\theta, \mathbf{v}) \mapsto x + (\theta, \mathbf{v}),$$

where  $B_{2d}(\mathbf{0}, \delta) \subseteq \mathbb{R}^{2d}$  is the open ball of center the origin and radius  $\delta$ . We then have:

1. the standard  $S^1$ -action is expressed by a translation in  $\theta$ ;
2.  $\mathbf{v} \in B_{2d}(\mathbf{0}, \delta) \mapsto m_x + \mathbf{v} =: \pi(x + (\theta, \mathbf{v}))$  ( $\theta$  being irrelevant) is a system of local coordinates on  $M$  centered at  $m_x$ ;
3.  $\gamma_x$  induces a unitary isomorphism  $T_x X \cong \mathbb{R} \oplus \mathbb{R}^{2d}$ , compatible with the decomposition of  $T_x X = V_x \oplus H_x$  as an orthogonal direct sum of the vertical and horizontal tangent space.
4. HLC can be locally and smoothly deformed with the base point  $x$ : for any  $x \in X$ , there exist an open neighborhood  $x \in X' \subseteq X$  and a  $\mathcal{C}^\infty$  map

$$\gamma : X' \times (-\pi, \pi) \times B_{2d}(\mathbf{0}, \delta) \rightarrow X,$$

such that  $\gamma_y(\theta, \mathbf{v}) =: \gamma(y, \theta, \mathbf{v})$  is a system of HLC centered at  $y$ , for each  $y \in X'$ .

One often writes  $x + \mathbf{v}$  for  $x + (0, \mathbf{v})$ .

In a HLCS, the universal nature of near-diagonal scaling asymptotics of certain kernels variously related to the Szegő kernel, such as the ones studied in this paper, is particularly transparent. In particular, these asymptotics generally involve a universal exponent, given by a quadratic function on  $\mathbb{R}^{2d} \times \mathbb{R}^{2d}$ , that we shall now define (following [BSZ] and [SZ]).

**Definition 1.11.** Let us define  $\psi_2 : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \rightarrow \mathbb{R}$  by setting

$$\psi_2(\mathbf{v}, \mathbf{w}) =: -i\omega_0(\mathbf{v}, \mathbf{w}) - \frac{1}{2} \|\mathbf{v} - \mathbf{w}\|^2,$$

where  $\omega_0$  is the standard symplectic structure, and  $\|\cdot\|$  is the standard Euclidean norm.

**Notation 1.2.** Given  $x \in X_\beta(\mathbf{s}_0)$ , and a choice of a HLC system centered at  $x$ , we shall let  $A = A_{m_x}$  be the corresponding unitary (i.e., symplectic and orthogonal) matrix representing  $d_x\phi_{\mathbf{s}_0}^X : T_{m_x}M \rightarrow T_{m_x}M$ . Then, since the action is holomorphic and Abelian,  $AJ_{m_x} = J_{m_x}A$  and  $A\xi_M(m_x) = \xi_M(m_x)$  for every  $\xi \in \mathfrak{t}$ . Therefore, we also have  $AJ_m(\xi_M(m_x)) = J_m(\xi_M(m_x))$ .

## 1.2 The statements

Our main result on the singularities of  $\text{tr}(\mathfrak{U})$  can be viewed euphemistically as a ‘directional local trace formula’. Before we state it, let us collect here all of our assumptions:

**General Hypothesis:** In the previous general setting, let us assume:

1.  $f_1, \dots, f_r : M \rightarrow \mathbb{R}$  are  $\mathcal{C}^\infty$ , Poisson commuting, and compatible with the Kähler structure  $(\omega, J)$  (Definition 1.1);
2. If  $\Phi =: (f_1, \dots, f_r)^t : M \rightarrow \mathfrak{t}^\vee$ , then  $\mathbf{0} \notin \Phi(M) \subseteq \mathfrak{t}^\vee$ ;
3.  $\mathbf{s}_0 \in \text{Per}(\phi^X)$ ;
4.  $\chi : \mathbb{R}^r \rightarrow \mathbb{R}$  is  $\mathcal{C}^\infty$  and compactly supported in a ball of center the origin and sufficiently small radius  $\epsilon > 0$  (although it’s unnecessary, we may assume that  $\chi, \widehat{\chi} \geq 0$ );
5.  $\chi_{\mathbf{s}_0}(\cdot) =: \chi(\cdot - \mathbf{s}_0)$  in (17);
6.  $\beta \in \mathfrak{t}^\vee$  has unit norm, and  $\beta \in \mathbb{R}_+ \cdot \Phi(m_x)$  for some  $x \in X(\mathbf{s}_0)$  (recall that  $m_x = \pi(x)$ ).

7.  $\Phi$  is transverse to  $\mathbb{R}_+ \cdot \beta$ .

*Remark 1.5.* As we have remarked, Condition 2 ensures that  $\Lambda_j \rightarrow \infty$  in  $\mathbb{R}^r$ , so that every joint eigenvalue has finite multiplicity (Lemma 2.1).

**Theorem 1.1.** *Assume that the General Hypothesis holds, and choose*

$$D > 0, \quad \delta \in (0, 1/2).$$

*Then, as  $\lambda \rightarrow +\infty$ , uniformly for*

$$\text{dist}_X(y, X_\beta(\mathbf{s}_0)) \geq D \lambda^{\delta-1/2}, \quad (21)$$

*(Definition 1.10) we have*

$$\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) = O(\lambda^{-\infty}).$$

Notice that the previous condition may be rewritten

$$\text{dist}_M(m_y, \pi(X_\beta(\mathbf{s}_0))) \geq D \lambda^{\delta-1/2},$$

where  $m_y =: \pi(y)$ , and that  $\pi(X_\beta(\mathbf{s}_0))$  is a union of connected components of  $M_\beta(\mathbf{s}_0)$ .

Theorem 1.1 shows that the asymptotics of  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)$  concentrate in a shrinking neighborhood of  $X_\beta(\mathbf{s}_0)$ ; this leads to considering appropriate scaling asymptotic near  $X_\beta(\mathbf{s}_0)$ , as we shall now make precise.

It will be proved in §2.1.5 that  $X_\beta(\mathbf{s}_0)$  is a submanifold of  $X$ , and its normal space at any  $x \in X_\beta(\mathbf{s}_0)$  splits naturally as an orthogonal direct sum

$$\begin{aligned} N_x(X_\beta(\mathbf{s}_0)) \\ \cong \ker(d_{m_x} \phi_{\mathbf{s}_0}^M - \text{id}_{T_{m_x} M})^\perp \oplus^\perp [J_{m_x} \circ \text{val}_{m_x}(\ker \Phi(m_x))]; \end{aligned} \quad (22)$$

here  $T_{m_x} M$  and its subspaces are viewed as vector subspaces of  $T_x X$ , by identifying  $T_{m_x} M$  with the horizontal tangent space  $H_x \subset T_x X$ .

In addition, as recalled in §1.1.7, in the neighborhood of any  $x_0 \in X_\beta(\mathbf{s}_0)$  we can find a smoothly varying family of HLCs's. Thus, locally near  $x_0$ , any  $z \in X$  within a distance  $D \lambda^{\delta-1/2}$  from  $X_\beta(\mathbf{s}_0)$  can be written  $z = x + \mathbf{v}$ , for unique  $x \in X_\beta(\mathbf{s}_0)$  and  $\mathbf{v} \in N_x(X_\beta(\mathbf{s}_0))$ , with  $\|\mathbf{v}\| \leq D' \lambda^{\delta-1/2}$  for some  $D' > 0$  (we may take  $D' = D + \epsilon'$  for any  $\epsilon' > 0$ ). In turn, in view of the previous direct sum decomposition we can also write

$$\mathbf{v} = \mathbf{w} + \mathbf{n} \quad (23)$$

where

$$\mathbf{w} \in \ker (d_{m_x} \phi_{\mathbf{s}}^M - \text{id}_{T_{m_x} M})^\perp \quad \text{and} \quad \mathbf{n} = J_{m_x}(\xi_M(m_x)) \quad (24)$$

with  $\xi \in \ker \Phi(m)$ ; here both  $\mathbf{w}$  and  $\xi$  are also uniquely determined, and both have norms  $O(\lambda^{\delta-1/2})$ .

In order to obtain the desired scaling asymptotics in a shrinking neighborhood of  $X_\beta(\mathbf{s}_0)$ , we shall replace the local parametrization  $y = x + \mathbf{v}$  by its rescaled version

$$y_\lambda =: x + \mathbf{v}/\sqrt{\lambda},$$

where now  $\|\mathbf{v}\| = O(\lambda^\delta)$ .

**Theorem 1.2.** *Assume that the General Hypothesis holds, and choose arbitrary constants*

$$D > 0, \quad \delta \in (0, 1/6).$$

*Then, uniformly in  $x \in X_\beta(\mathbf{s}_0)$  and  $\mathbf{v} = \mathbf{w} + \mathbf{n} \in N_x(X_\beta(\mathbf{s}_0))$  as in (23) and (24) with  $\|\mathbf{v}\| \leq D\lambda^\delta$ , the following asymptotic expansion holds:*

$$\begin{aligned} & \mathcal{S}_\chi(\lambda\beta, \mathbf{s}_0, y_\lambda, y_\lambda) \\ & \sim \frac{2^{\frac{r+1}{2}} \pi}{\|\Phi(m_x)\|} \cdot \left( \frac{\lambda}{\pi \|\Phi(m_x)\|} \right)^{d+\frac{1-r}{2}} \frac{e^{-i\lambda\langle\beta, \mathbf{s}_0\rangle}}{\mathcal{D}(m)} e^{[\psi_2(A\mathbf{w}, \mathbf{w}) - 2\|\mathbf{n}\|^2]/\|\Phi(m_x)\|} \\ & \cdot \sum_{\ell \geq 0} \lambda^{-\ell/2} \mathcal{R}_\ell(x; \mathbf{n}, \mathbf{w}), \end{aligned} \quad (25)$$

where  $\psi_2$  is as in Definition 1.11,  $A$  is defined in Notation 1.2 in §1.1.7, and  $\mathcal{R}_\ell(x; \cdot, \cdot)$  is a polynomial of degree  $\leq 3\ell$ , and parity  $(-1)^\ell$ . We have  $\mathcal{R}_0 = \chi(\mathbf{0})$ .

Notice that, with  $\mathbf{w}$  and  $\mathbf{n}$  as in (24), we have for some constant  $a > 0$

$$\begin{aligned} \Re(\psi_2(A\mathbf{w}, \mathbf{w}) - 2\|\mathbf{n}\|^2) &= -\frac{1}{2} \|A\mathbf{w} - \mathbf{w}\|^2 - 2\|\mathbf{n}\|^2 \\ &\leq -a (\|\mathbf{w}\|^2 + \|\mathbf{n}\|^2); \end{aligned}$$

therefore (25) describes, in rescaled coordinates, an exponential decrease of  $\mathcal{S}_\chi(\lambda\beta, \mathbf{s}_0, y_\lambda, y_\lambda)$  along normal directions to  $X_\beta(\mathbf{s}_0)$ .

Inserting the local asymptotics in Theorem 1.2 in (19) we obtain a global asymptotic expansion for  $\mathcal{F}(\chi_{\mathbf{s}_0} \cdot \text{tr}(\mathfrak{U}))(\lambda\beta)$ . Before stating this, we need to introduce a further Poincaré type invariant. Given  $m \in M(\mathbf{s}_0)$ , let

$$\kappa_{m, \mathbf{s}_0} : N_m(M(\mathbf{s}_0)) \rightarrow N_m(M(\mathbf{s}_0))$$

be the restriction of  $\text{id}_{T_m M} - d_m \phi_{-\mathbf{s}_0}^M$ . Then  $\kappa_{m, \mathbf{s}_0}$  is an automorphism, and its determinant

$$\mathfrak{k}(m, \mathbf{s}_0) =: \det(\kappa_{m, \mathbf{s}_0}) \quad (26)$$

is locally constant on  $M(\mathbf{s}_0)$ .

**Definition 1.12.** Let  $M_\beta(\mathbf{s}_0)_j$ ,  $1 \leq j \leq b$ , where  $b = b(\beta, \mathbf{s}_0)$ , denote the connected components of  $\pi(X_\beta(\mathbf{s}_0))$ ; these are some of the connected components of  $M_\beta(\mathbf{s}_0)$ , but perhaps not all of them. We shall denote by  $\mathbf{c}_j(\mathbf{s}_0) \in \mathbb{C}$  the constant value of  $\mathfrak{k}(m, \mathbf{s}_0)$  on  $M_\beta(\mathbf{s}_0)_j$ . Also, let  $c_j$  (respectively,  $f_j =: d - c_j$ ) be the complex codimension (respectively, complex dimension) of  $M(\mathbf{s}_0)$  in  $M$  along  $M_\beta(\mathbf{s}_0)_j$ , that is, the complex codimension (respectively, dimension) in  $M$  of the unique connected component of  $M(\mathbf{s}_0)$  containing  $M_\beta(\mathbf{s}_0)_j$ .

**Corollary 1.2.** *Under the same assumptions as in Theorem 1.2, and with the notation (14), we have*

$$\mathcal{F}(\chi_{\mathbf{s}_0} \cdot \mathrm{tr}(\mathfrak{U}))(\lambda \beta) = \sum_{j=1}^b \mathcal{F}_j(\chi_{\mathbf{s}_0} \cdot \mathrm{tr}(\mathfrak{U}))(\lambda \beta),$$

where each summand admits an asymptotic expansion

$$\mathcal{F}_j(\chi_{\mathbf{s}_0} \cdot \mathrm{tr}(\mathfrak{U}))(\lambda \beta) \sim \frac{2\pi}{\mathbf{c}_j(\mathbf{s}_0)} e^{-i\lambda \langle \beta, \mathbf{s}_0 \rangle} \left( \frac{\lambda}{\pi} \right)^{f_j+1-r} \cdot \sum_{k \geq 0} \lambda^{-k} \mathcal{U}_{jk}(\mathbf{s}_0, \beta),$$

with the leading order term given by

$$\mathcal{U}_{j0}(\mathbf{s}_0, \beta) =: \chi(\mathbf{0}) \cdot \int_{M_\beta(\mathbf{s}_0)_j} \frac{1}{\|\Phi(m)\|^{f_j+2-r}} \frac{1}{\mathcal{D}(m)} dV_{M_\beta(\mathbf{s}_0)_j}(m).$$

Here  $dV_{M_\beta(\mathbf{s}_0)_j}$  is the Riemannian volume density on  $M_\beta(\mathbf{s}_0)_j$ .

*Remark 1.6.* For the case  $r = 1$ , see [P3].

## 2 Preliminaries

### 2.1 The moment map

In the following, let  $\phi^M : \mathbb{R}^r \times M \rightarrow M$  be an Hamiltonian and holomorphic action, with moment map  $\Phi = (f_1, \dots, f_r)^t : M \rightarrow \mathbb{R}^r$ , such that  $\mathbf{0} \notin \Phi(M)$ .

#### 2.1.1 $\Lambda_j \rightarrow \infty$ and $\mathrm{tr}(\mathfrak{U})$ as a temperate distribution

Our first remark is that under the given assumption the joint eigenvalues  $\Lambda_j$  drift to infinity in  $\mathbb{R}^r$  as  $j \rightarrow +\infty$ :

**Lemma 2.1.** *Given that  $\mathbf{0} \notin \Phi(M)$ , we have  $\Lambda_j \rightarrow \infty$  as  $j \rightarrow +\infty$ .*

*Proof of Lemma 2.1.* The self-adjoint first order Toeplitz operator  $\mathfrak{T}_k$  has eigenvalue  $\lambda_{kj} \in \mathbb{R}$  on  $e_j$ . Therefore, the second order Toeplitz operator  $\mathfrak{T}_k^2 \geq 0$  has eigenvalue  $\lambda_{jk}^2$  on  $e_j$ ; its principal symbol is

$$\sigma_{\mathfrak{T}_k^2}((x, r\alpha_x)) = \sigma_{\mathfrak{T}_k}((x, r\alpha_x))^2 = r^2 f_k(m_x)^2.$$

Let us define

$$\|\mathfrak{T}\| =: \left( \sum_{k=1}^r \mathfrak{T}_k^2 \right)^{1/2}. \quad (27)$$

Then  $\|\mathfrak{T}\|$  is a first order Toeplitz operator, with eigenvalue  $\|\Lambda_j\|$  on  $e_j$ ; by the theory of [BtG] and the corresponding results for pseudodifferential operators [Se], [Sh], its principal symbol is

$$\sigma_{\|\mathfrak{T}\|}(x, r\alpha_x) = r \left( \sum_{k=1}^r f_k(m_x)^2 \right)^{1/2} = r \|\Phi(m_x)\| > 0.$$

It follows that  $\|\Lambda_j\| \rightarrow +\infty$  as  $j \rightarrow +\infty$  [BtG].

□

**Corollary 2.1.** *Given that  $0 \notin \Phi(M)$ , every joint eigenvalue has finite multiplicity.*

Thus there exists  $j_0$  such that  $\Lambda_j \neq 0$  for  $j \geq j_0$ . We can strengthen the previous statement as follows:

**Lemma 2.2.** *If  $a > 0$  is sufficiently large, then*

$$\sum_{j \geq j_0} \|\Lambda_j\|^{-a} < +\infty.$$

*Proof of Lemma 2.2.* Let  $\|\mathfrak{T}\|$  be as in (27). We can assume that  $\|\mathfrak{T}\|$  is the restriction to  $H(X)$  of a first-order self-adjoint pseudodifferential operator  $Q$ , with everywhere positive principal symbol, and commuting with  $\Pi$  [BtG]. If  $\eta_1 \leq \eta_2 \leq \dots$  is the sequence of the eigenvalues of  $Q$ , repeated according to multiplicity, we then have  $\eta_l > 0$  for  $l \geq l_1$  for some appropriate  $l_1 \gg 0$ , and

$$\sum_{l \geq l_1} \eta_l^{-a} < +\infty$$

for every  $a \gg 0$  (Theorem 12.2 of [GrSj]). Since the  $\|\Lambda_j\|$ 's are the eigenvalues of the restriction of  $Q$  to  $H(X)$ , they form a subsequence of the  $\eta_l$ 's, and the statement follows.

□



**Corollary 2.2.**  $\sum_j \delta_{-\Lambda_j}$  and

$$\mathrm{tr}(\mathfrak{U}) = \sum_j e^{i\langle \Lambda_j, \cdot \rangle} = \mathcal{F} \left( \sum_j \delta_{-\Lambda_j} \right)$$

are temperate distribution on  $\mathbb{R}^r$ .

### 2.1.2 An intrinsic vector field

For every  $m \in M$  there is a unique  $\Xi(m) \in \mathfrak{t}$  such that  $\Xi(m) \in \ker \Phi(m)^\perp$ ,  $\|\Xi(m)\| = 1$  (with respect to the standard Euclidean product), and

$$\langle \Phi(m), \Xi(m) \rangle = \|\Phi(m)\|.$$

Equivalently, if  $\eta(m) \in \mathfrak{t}$  corresponds to  $\Phi(m) \in \mathfrak{t}^\vee$  under the isomorphism  $\mathfrak{t} \cong \mathfrak{t}^\vee$  induced by the standard Euclidean product, then

$$\Xi(m) = \eta(m) / \|\eta(m)\| = \eta(m) / \|\Phi(m)\|.$$

We thus obtain a  $\mathcal{C}^\infty$  map  $m \in M \mapsto \Xi(m) \in \mathfrak{t}$ , taking value in the unit sphere, and a vector field  $V \in \mathfrak{X}(M)$  on  $M$ , intrinsically associated to  $\Phi$ , given by

$$V(m) =: \Xi(m)_M(m) \quad (m \in M),$$

in the notation of Definition 1.6.

Furthermore, given  $\nu \in \mathfrak{t}$  and  $m \in M$  we have a unique orthogonal decomposition

$$\nu = \nu'(m) + a(\nu, m) \Xi(m), \tag{28}$$

where  $\nu'(m) \in \ker \Phi(m)$ , and  $a(\nu, m) = \langle \nu, \Xi(m) \rangle_{\mathfrak{t}}$ .

### 2.1.3 Transversality

Given that  $\Phi(m) \neq \mathbf{0}$  for every  $m \in M$ , we obtain a  $\mathcal{C}^\infty$  map to the unit sphere:

$$\Phi_u =: \frac{1}{\|\Phi\|} \Phi : M \rightarrow S^{r-1} \subseteq \mathfrak{t}^\vee \cong \mathbb{R}^r.$$

If  $\beta \in S^{r-1} \subseteq \mathfrak{t}^\vee$  (the unit sphere), we have set  $M_\beta =: \Phi^{-1}(\mathbb{R}_+ \beta)$  e  $X_\beta =: \pi^{-1}(M_\beta)$ . Clearly,  $M_\beta = \Phi_u^{-1}(\beta)$ .

**Lemma 2.3.** *Consider  $\beta \in \mathfrak{t}^\vee$  of unit norm. Under the previous assumptions, the following conditions are equivalent:*

1.  $\Phi$  is transverse to  $\mathbb{R}_+ \beta$ ;
2.  $\phi^X$  is locally free on  $X_\beta$ , that is, the stabilizer subgroup in  $\mathbb{R}^r$  of any  $x \in X_\beta$  is discrete;
3. for any  $x \in X_\beta$ ,  $\text{val}_x : \mathfrak{t} \rightarrow T_x X$ ,  $\xi \mapsto \xi_X(x)$ , is injective;
4. for any  $m \in M_\beta$ , the restriction of the evaluation,  $\text{val}_m : \ker \Phi(m) \rightarrow T_m M$ , is injective;
5.  $\beta$  is a regular value of  $\Phi_u$ .

The first four points follow from the discussion in §2.2 of [P4], and the latter is straightforward.

**Corollary 2.3.** *Given that  $\Phi$  is transverse to  $\mathbb{R}_+ \cdot \beta$ , there is a  $\phi^X$ -invariant tubular neighborhood  $X' \subseteq X$  of  $X_\beta$  on which  $\phi^X$  is locally free.*

**Corollary 2.4.** *Given that  $M$  is compact and that  $\mathbf{0} \notin \Phi(M)$ , if  $\beta \in \mathfrak{t}^\vee$  has unit norm and  $\Phi$  is transverse to  $\mathbb{R}_+ \cdot \beta$ , then there exists a constant  $C > 0$  such that for all  $m \in M$  we have*

$$\text{dist}_{\mathfrak{t}^\vee}(\mathbb{R}_+ \cdot \Phi(m), \beta) \geq C \text{dist}_M(m, M_\beta).$$

**Corollary 2.5.** *Under the assumptions of Corollary 2.4, there exists a constant  $C > 0$  such that for all  $m \in M$  and  $t > 0$  we have*

$$\|t \Phi(m) - \beta\| \geq C \text{dist}_M(m, M_\beta)$$

where  $\|\cdot\|$  is the standard Euclidean norm on  $\mathfrak{t}^\vee \cong \mathbb{R}^r$ .

#### 2.1.4 Transversality and locally isolated periods

We are interested in the diagonal asymptotics of (18), and as we shall see these are non trivial only in the vicinity of  $X_\beta(\mathbf{s}_0)$ .

In general, periods of  $\phi^X$  needn't be isolated; nonetheless, under the previous transversality assumptions,  $\mathbf{s}_0$  is indeed an isolated period in a neighborhood of  $X_\beta(\mathbf{s}_0)$ . Let us formalize this point by giving first a definition.

**Definition 2.1.** Suppose  $\mu : G \times P \rightarrow P$  is a  $\mathcal{C}^\infty$  action of a Lie group  $G$  on a manifold  $P$ , and let  $P' \subseteq P$  be a  $G$ -invariant open subset. For any  $g \in G$ , let  $P(g) \subseteq P$  be the fixed locus of  $\mu_g : P \rightarrow P$ . We shall say that  $g_0 \in G$  is an isolated period of  $\mu$  on  $P'$  if  $P(g_0) \cap P' \neq \emptyset$  and there exists an open neighborhood  $G' \subseteq G$  of  $g_0$  such that  $P(g) \cap P' = \emptyset$  for all  $g \in G' \setminus \{g_0\}$ . If  $p \in P(g_0)$ , we shall say that  $g_0$  is a locally isolated period at  $p$  if it is an isolated period on  $P'$  for some open  $\mu$ -invariant neighborhood  $P'$  of  $p$ .

**Proposition 2.1.** *Let  $G$  be an Abelian Lie group,  $(P, \varphi)$  a Riemannian manifold, and  $\mu : G \times P \rightarrow P$  a  $C^\infty$  action of  $G$  on  $P$  as a group of Riemannian isometries. Suppose that  $p_0 \in P$  and that  $\mu$  is locally free at  $p_0$  (i.e.,  $\xi_P(p_0) \neq 0 \in T_{p_0}P$ , for all  $\xi \in \mathfrak{g}$  with  $\xi \neq \mathbf{0}$ , where  $\mathfrak{g}$  is the Lie algebra of  $G$ ). Consider  $g_0 \in G$  with  $\mu_{g_0}(p_0) = p_0$ ; then  $g_0$  is a locally isolated period of  $\mu$  at  $p_0$ .*

The statement is quite straightforward when the action is proper. In fact, the hypothesis implies that the stabilizer subgroup  $\text{St}(p_0) \subseteq G$  of  $p_0$  is discrete, whence there exists an open neighborhood  $G' \subseteq G$  of  $g_0$ , such that  $G' \cap \text{St}(p_0) = \{g_0\}$ . On the other hand, since the action is proper there exists an invariant open neighborhood  $P' \subseteq P$  such that  $\text{St}(p) \subseteq \text{St}(p_0)$  for every  $p' \in P'$  (see e.g. Appendix B of [GKK]), and this implies the statement. The claim follows, therefore, whenever  $\phi^X$  descends to an action of the compact torus  $\mathbb{T}^r = \mathbb{R}^r/\mathbb{Z}^r$ . Since we do not wish to impose this condition, we give a general proof.

*Proof of Proposition 2.1.* Let  $d_P$  and  $d_G$  denote the dimensions of  $P$  and  $G$ , respectively. Let us fix some Euclidean scalar product on  $\mathfrak{g}$ , and let  $B_{\mathfrak{g}}(\mathbf{0}, \delta) \subseteq \mathfrak{g}$  be the ball centered at the origin and of radius  $\delta > 0$ . To abridge notation, let us set  $g \cdot p =: \mu(g, p)$ .

Then for sufficiently small  $\delta$  the map

$$\gamma : B_{\mathfrak{g}}(\mathbf{0}, \delta) \longrightarrow P, \quad \xi \mapsto e^\xi \cdot p_0$$

is a diffeomorphism onto its image; here  $\xi \mapsto e^\xi$  is of course the exponential map on  $G$ .

Thus  $Q(\delta) =: \gamma(B_{\mathfrak{g}}(\mathbf{0}, \delta)) \subseteq P$  is a smooth  $d_G$ -dimensional submanifold passing through  $p_0$ . Let  $N \subseteq TP|_Q$  be Riemannian normal bundle to  $Q$  and let  $N(\epsilon) \subseteq N$  be the  $\epsilon$ -neighborhood of the zero section for some  $\epsilon > 0$  sufficiently small. Also, let  $N(\delta, \epsilon)$  be the pull-back of  $N(\epsilon)$  to  $B_{\mathfrak{g}}(\mathbf{0}, \delta)$ . Thus, perhaps after passing to smaller values of  $\delta$  and  $\epsilon$  if necessary, the normal exponential map provides a smooth map

$$\tilde{\gamma} : N(\delta, \epsilon) \longrightarrow P, \quad (\xi, \mathbf{n}) \mapsto \exp_P(e^\xi \cdot p_0, \mathbf{n}),$$

where  $\exp_P$  is the exponential map of  $P$ , defined on some open neighborhood of the zero section in  $TP$ .

Then, again perhaps after passing to smaller  $\epsilon, \delta$  if necessary,  $\tilde{\gamma}$  is a diffeomorphism onto its image  $R(\delta, \epsilon)$ , which is an open tubular neighborhood of  $Q(\delta)$ .

Now let  $R' = R(\delta', \epsilon')$  be similarly constructed, but with suitably smaller  $\delta', \epsilon' > 0$ . Consider  $r = \exp_P(e^\xi \cdot p_0, \mathbf{n}) \in R'$ , and  $g = g_0 e^\eta$  with  $\eta \in$

$B_g(\mathbf{0}, \delta')$ , and suppose  $r \in P(g)$ . Then, because  $G$  is Abelian and acts as a group of Riemannian isometries we have

$$\begin{aligned} g \cdot r &= g_0 e^\eta \cdot \exp_P(e^\xi \cdot p_0, \mathbf{n}) \\ &= \exp_P(e^{\xi+\eta} \cdot p_0, d_{e^\xi \cdot p_0} \mu_{g_0 e^\eta}(\mathbf{n})) \\ &= r = \exp_P(e^\xi \cdot p_0, \mathbf{n}). \end{aligned}$$

This forces however  $e^{\xi+\eta} \cdot p_0 = e^\xi \cdot p_0$ , whence  $\eta = 0$  and so  $g = g_0$ . Thus  $g_0$  is an isolated period of  $\mu$  on  $R'$ .

Now let  $R'' =: G \cdot R'$  be the  $\mu$ -saturation of  $R'$ . Since  $G$  is Abelian, if  $r'' \in R''$  and  $r'' = g \cdot r'$  for some  $r' \in R'$ , then  $r'$  and  $r''$  have the same stabilizer. Therefore,  $R''$  is an invariant open neighborhood of  $p_0$ , and  $g_0$  is an isolated period of  $\mu$  on  $R''$ .  $\square$

**Corollary 2.6.** *Under the assumptions of Corollary 2.3, there is a  $\phi^X$ -invariant neighborhood  $X'$  of  $X_\beta(\mathbf{s}_0)$  on which  $\mathbf{s}_0$  is an isolated period.*

*Proof of Corollary 2.6.* By Corollary 2.3 and Proposition 2.1, every  $x \in X_\beta(\mathbf{s}_0)$  has an invariant neighborhood  $X'_x$  on which  $\mathbf{s}_0$  is an isolated period. By compactness of  $X_\beta(\mathbf{s}_0)$ , we may find finitely many such neighborhoods, say  $X'_1, \dots, X'_k$ , whose union  $X'$  contains  $X_\beta(\mathbf{s}_0)$ , and such that  $\mathbf{s}_0$  is the only period of  $\phi^X$  on  $X'_j$  contained in  $B_{\mathbb{R}^r}(\mathbf{s}_0, \delta_j)$  for some  $\delta_j > 0$ . Then  $X'$  is invariant and  $\mathbf{s}_0$  is the only period on  $X'$  in  $B_{\mathbb{R}^r}(\mathbf{s}_0, \delta)$ , where  $\delta = \min(\delta_j)$ .  $\square$

### 2.1.5 Transversality and fixed loci

Since  $\phi_s^M : M \rightarrow M$  is holomorphic and symplectic,  $M(\mathbf{s})$  is a (compact) complex submanifold of  $M$  (Definition 1.9), and its tangent space at any  $m \in M(\mathbf{s})$  is

$$T_m M(\mathbf{s}) = \ker(d_m \phi_s^M - \text{id}_{T_m M}), \quad (29)$$

a complex subspace of  $T_m M$ .

In particular, since  $\mathbb{R}^r$  is an Abelian Lie group we have for any  $\xi \in \mathfrak{t}$  and  $m \in M(\mathbf{s})$  that

$$\xi_M(m), J_m(\xi_M(m)) \in \ker(d_m \phi_s^M - \text{id}_{T_m M}). \quad (30)$$

On the other hand, if  $\beta \in \mathfrak{t}^\vee$ ,  $\beta \neq \mathbf{0}$  and  $\Phi$  is transverse to  $\mathbb{R}_+ \cdot \beta$ , then  $M_\beta$  is a (real) compact submanifold of  $M$ , of codimension  $r - 1$ ; for any  $m \in M_\beta$ , by the discussion in [P4] the normal bundle  $N_m(M_\beta)$  to  $M_\beta$  at  $m$  is given by

$$N_m(M_\beta) = J_m \circ \text{val}_m(\ker \Phi(m)) \subseteq T_m M. \quad (31)$$

Since  $X_\beta$  is a union of connected components of  $\pi^{-1}(M_\beta)$ , its normal space  $N_x(X_\beta)$  at any  $x \in X_\beta$  is the horizontal lift of the normal space  $N_{m_x}(M_\beta)$ , where  $m_x = \pi(x)$ . In view of Remark 1.1, this is

$$N_x(X_\beta) = N_{m_x}(M_\beta)^\sharp = \text{val}_x(\ker \Phi(m_x)). \quad (32)$$

**Lemma 2.4.** *Suppose as above that  $\Phi : M \rightarrow \mathfrak{t}$  is transverse to  $\mathbb{R}_+ \cdot \beta$ . Then for any  $\mathbf{s} \in \mathbb{R}^r$  the following holds:*

1.  $M(\mathbf{s})$  and  $M_\beta$  are transverse submanifolds of  $M$ ;
2. for any  $m \in M_\beta(\mathbf{s}) =: M(\mathbf{s}) \cap M_\beta$ , the normal bundle to  $M_\beta(\mathbf{s})$  at  $m$  is the orthogonal direct sum

$$N_m(M_\beta(\mathbf{s})) = \ker(d_m \phi_{\mathbf{s}}^M - \text{id}_{T_m M})^\perp \oplus^\perp [J_m \circ \text{val}_m(\ker \Phi(m))].$$

*Proof of Lemma 2.4.* We need to show that  $T_m M = T_m M(\mathbf{s}) + T_m M_\beta$  for any  $m \in M(\mathbf{s}) \cap M_\beta$ , and this is equivalent to  $N_m M(\mathbf{s}) \cap N_m M_\beta = (0)$ . In view of (29), (30) and (31), this is

$$\begin{aligned} N_m M(\mathbf{s}) \cap N_m M_\beta &= \ker(d_m \phi_{\mathbf{s}}^M - \text{id}_{T_m M})^\perp \cap J_m \circ \text{val}_m(\ker \Phi(m)) \\ &\subseteq \ker(d_m \phi_{\mathbf{s}}^M - \text{id}_{T_m M})^\perp \cap \ker(d_m \phi_{\mathbf{s}}^M - \text{id}_{T_m M}) = (0). \end{aligned}$$

Therefore,  $M(\mathbf{s}, \beta)$  is a submanifold of  $M$ , and at any  $m \in M(\mathbf{s}, \beta)$  we have  $T_m M(\mathbf{s}, \beta) = T_m M(\mathbf{s}) \cap T_m M(\beta)$ . Thus the normal bundle is  $N_m M(\mathbf{s}, \beta) = N_m M(\mathbf{s}) + N_m M(\beta)$ , and the inclusion above also shows that this is an orthogonal direct sum. □

## 2.2 $\mathcal{U}$ and the singularities of its trace

### 2.2.1 $\mathfrak{U}$ as a complex FIO

In the present compatible setting, the operator  $\mathfrak{U}(\mathbf{s})$  in (9) has a simple expression in terms of  $\phi_{\mathbf{s}}^X$  and the Szegő projector  $\Pi$ . Namely, let  $(e_j)$  be a complete orthonormal system of  $H(X)$ , so that

$$\Pi(x, y) = \sum_j e_j(x) \cdot \overline{e_j(y)}.$$

Then the distributional kernel (11) of  $\mathfrak{U}(\mathbf{s}) = (\phi_{-\mathbf{s}}^X)^* \circ \Pi$  may also be expressed as

$$\mathfrak{U}(\mathbf{s}, x, y) = \sum_j e_j(\phi_{-\mathbf{s}}^X(x)) \cdot \overline{e_j(y)} = \Pi(\phi_{-\mathbf{s}}^X(x), y). \quad (33)$$

Therefore, since the singular support of  $\Pi$  is the diagonal [F], the singular support of  $\mathfrak{U}(\mathbf{s})$  is the graph of  $\phi_{-\mathbf{s}}^X$ .

In addition, by [BtSj] near the diagonal we have a microlocal description of  $\Pi$  as an FIO with complex phase, of the form

$$\Pi(x, y) \sim \int_0^{+\infty} e^{it\psi(x, y)} s(t, x, y) dt, \quad (34)$$

where  $\Im\psi \geq 0$ , and  $s(t, x, y) \sim \sum_{j \geq 0} t^{d-j} s_j(x, y)$  (see also the discussions in [Z], [SZ]). Thus, near the graph of  $\phi_{-\mathbf{s}}^X$  we have with  $x_{\mathbf{s}} =: \phi_{-\mathbf{s}}^X(x)$

$$\mathfrak{U}(\mathbf{s}, x, y) \sim \int_0^{+\infty} e^{it\psi(x_{\mathbf{s}}, y)} s(t, x_{\mathbf{s}}, y) dt. \quad (35)$$

*Remark 2.1.* With  $\psi$  as in (34), one has  $d_{(x, x)}\psi = (\alpha_x, -\alpha_x)$  for any  $x \in X$ , and more generally for any  $\vartheta \in \mathbb{R}$

$$d_{(e^{i\vartheta} x, x)}\psi = (e^{i\vartheta} \alpha_{e^{i\vartheta} x}, -e^{i\vartheta} \alpha_x).$$

*Remark 2.2.* As shown in §3 of [SZ], in a system of HLC centered at  $x \in X$ , the phase  $t\psi$  satisfies the following expansion:

$$\begin{aligned} t\psi(x + (\theta, \mathbf{v}), x + \mathbf{w}) \\ = it [1 - e^{i\theta}] - it\psi_2(\mathbf{v}, \mathbf{w}) e^{i\theta} + tR_3(\mathbf{v}, \mathbf{w}) e^{i\theta}, \end{aligned}$$

where  $\psi_2$  is as in Definition 1.11, while  $R_3 : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \rightarrow \mathbb{C}$  is  $\mathcal{C}^\infty$  and vanishes to third order at the origin.

The description of  $\Pi$  as an FIO in (34), in view of Corollary 1.3 of [BtSj] and Remark 2.1 above, implies, as is well-known, that the wave front of  $\Pi$  is

$$\text{WF}(\Pi) = \{((x, x), r(\alpha_x, -\alpha_x)) : r > 0, x \in X\} \subseteq T^*(X \times X).$$

It follows that the wave front of  $\mathfrak{U}(\mathbf{s})$  is

$$\text{WF}(\mathfrak{U}(\mathbf{s})) = \{((x_{\mathbf{s}}, x), r(\alpha_{x_{\mathbf{s}}}, -\alpha_x)) : r > 0, x \in X\} \subseteq T^*(X \times X). \quad (36)$$

We can view  $\mathfrak{U}$  as an operator  $\mathcal{C}^\infty(\mathbb{R}^r \times X) \rightarrow \mathcal{C}^\infty(X)$ , with distributional kernel  $\mathfrak{U} \in \mathcal{D}(\mathbb{R}^r \times X \times X \times X)$ ; given (35) and Remark 2.1, its wave front is

$$\begin{aligned} \text{WF}(\mathfrak{U}) = \left\{ \left( (\mathbf{s}, x, x_{\mathbf{s}}), r(\Phi(m_x), \alpha_x, -\alpha_{x_{\mathbf{s}}}) \right) : \right. \\ \left. \mathbf{s} \in \mathbb{R}^r, x \in X, r > 0 \right\}, \end{aligned} \quad (37)$$

where  $m_x =: \pi(x)$ ; we have used that  $\alpha$  is  $\phi^X$ -invariant.

### 2.2.2 Functorial description of $\text{tr}(\mathfrak{U})$

Since we have chosen a volume form on  $X$ , there are naturally induced volume forms (whence densities and half-densities) on  $\mathbb{R}^r \times X$  and  $\mathbb{R}^r \times X \times X$ ; in terms of the latter, we may extend the pull-back operation of functions under  $\mathcal{C}^\infty$  maps involving these manifolds to  $\mathcal{C}^\infty$  densities. Similarly, the push-forward operation, which by duality is naturally defined on densities under proper  $\mathcal{C}^\infty$  maps, extends with the given choices to  $\mathcal{C}^\infty$  functions. In addition, these functorial operations may be extended continuously to generalized densities, as far as the appropriate conditions involving wave fronts are met ([H2], [D]). The identification between functions, densities and half-densities will be left implicit in the following.

Let us then consider the diagonal map  $\Delta : \mathbb{R} \times X \rightarrow \mathbb{R} \times X \times X$ ,  $(\mathbf{s}, x) \mapsto (\mathbf{s}, x, x)$ . In view of (37) and the condition  $\Phi(m) \neq \mathbf{0} \forall m \in M$ , the pull-back

$$\Delta^*(\mathfrak{U}) = \sum_{j=1}^{+\infty} e^{i\langle \Lambda_j, \mathbf{s} \rangle} e_j(x) \cdot \overline{e_j(x)} \in \mathcal{D}'(\mathbb{R}^r \times X)$$

is well-defined; by (37) and the functorial properties of wave fronts (see [H2] and [D]), it has wave front

$$\begin{aligned} \text{WF}(\Delta^*(\mathfrak{U})) &= \left\{ \left( (\mathbf{s}, x), r(\Phi(m_x), 0) \right) : \right. \\ &\quad \left. \mathbf{s} \in \mathbb{R}^r, x \in \text{Fix}(\phi_{\mathbf{s}}^X), r > 0 \right\}. \end{aligned} \quad (38)$$

where  $\text{Fix}(\phi_{\mathbf{s}}^X) = \{x \in X : x = x_{\mathbf{s}}\}$ .

Moreover, since the projection  $p : \mathbb{R}^r \times X \rightarrow X$  is proper, the push-forward  $p_*(\Delta^*(\mathfrak{U})) \in \mathcal{D}'(\mathbb{R}^r)$  is also well-defined, and by orthonormality of the  $e_j$ 's we have

$$p_*(\Delta^*(\mathfrak{U})) = \sum_{j=1}^{+\infty} e^{i\langle \Lambda_j, \mathbf{s} \rangle} = \text{tr}(\mathfrak{U}).$$

In addition, given (38) its wave front is

$$\begin{aligned} \text{WF}(\text{tr}(\mathfrak{U})) &= \left\{ \left( \mathbf{s}, r\Phi(m_x) \right) : \mathbf{s} \in \mathbb{R}^r, x \in \text{Fix}(\phi_{\mathbf{s}}^X), r > 0 \right\} \\ &= \bigcup_{\mathbf{s} \in \mathbb{R}^r} \{\mathbf{s}\} \times \text{WF}(\text{tr}(\mathfrak{U}))_{\mathbf{s}}, \end{aligned} \quad (39)$$

where for each  $\mathbf{s} \in \mathbb{R}^r$  we have set

$$\text{WF}(\text{tr}(\mathfrak{U}))_{\mathbf{s}} =: \bigcup_{x \in \text{Fix}(\phi_{\mathbf{s}}^X)} \mathbb{R}_+ \cdot \Phi(m_x) = \bigcup_{x \in \text{Fix}(\phi_{\mathbf{s}}^X)} \mathbb{R}_+ \cdot \Phi_{\mathbf{u}}(m_x).$$

This proves Proposition 1.1 and Corollary 1.1.

We are interested in estimating the asymptotics of (15). In view of the above we have:

**Corollary 2.7.** *Under the previous assumptions, suppose*

$$(\mathbf{s}_0, \beta_0) \in T^*(\mathbb{R}^r) \setminus \text{WF}(\text{tr}(\mathfrak{U})), \quad \|\beta_0\| = 1.$$

*Then there exists  $\epsilon > 0$  such that for every  $\chi \in \mathcal{C}_0^\infty(B_r(\mathbf{0}, \epsilon))$  we have*

$$\langle \text{tr}(\mathfrak{U}), \chi_{\mathbf{s}_0} e^{-i\lambda \langle \beta, \cdot \rangle} \rangle = O(\lambda^{-\infty})$$

*uniformly in  $\beta \in \mathbb{R}^r$  with  $\|\beta\| = 1$ ,  $\|\beta - \beta_0\| < \epsilon$ .*

Here  $B_r(\mathbf{0}, \epsilon) \subseteq \mathbb{R}^r$  is the open ball of center the origin and radius  $\epsilon$ , while  $\chi_{\mathbf{s}_0}(\mathbf{s}) = \chi(\mathbf{s} - \mathbf{s}_0)$ .

### 2.2.3 The smoothing operator

As in the case  $r = 1$ , the operators  $\mathfrak{U}(\mathbf{s})$  may be averaged with a weight of rapid decrease to obtain a smoothing operator.

**Lemma 2.5.** *For any  $\chi \in \mathcal{S}(\mathbb{R}^r)$ , the operator*

$$S_\chi =: \int_{\mathbb{R}^r} \chi(\mathbf{s}) \mathfrak{U}(\mathbf{s}) \, d\mathbf{s}$$

*is smoothing, and its kernel  $S_\chi(\cdot, \cdot) \in \mathcal{C}^\infty(X \times X)$  is given by*

$$S_\chi(x, y) = \sum_j \widehat{\chi}(-\Lambda_j) e_j(x) \cdot \overline{e_j(y)}. \quad (40)$$

The following is an adaptation of an argument in §12 of [GrSj].

*Proof of Lemma 2.5.* Let  $Q$  be as in the proof of Lemma 2.2. Since the  $e_j$ 's are orthonormal eigenfunctions of  $Q$ , with eigenvalues  $\|\Lambda_j\|$ , a standard argument based on the Sobolev inequalities shows that for some fixed  $j_0$  and all  $j \geq j_0$  we have

$$\|e_j\|_{\mathcal{C}^k} \leq C_k \|\Lambda_j\|^{k+2d+1}.$$

Since  $\widehat{\chi} \in \mathcal{S}(\mathbb{R}^r)$ , for any  $N > 0$  there exists  $C_N > 0$  such that for all  $j \geq j_0$  we have

$$|\widehat{\chi}(-\Lambda_j)| \leq C'_N \|\Lambda_j\|^{-N}.$$

Thus Lemma 2.2 implies that (40) converges in  $\mathcal{C}^\infty(X \times X)$ . Given this, that (40) is indeed the distributional kernel of  $S_\chi$  follows by first applying it to finite linear combinations of the  $e_j$ 's, and then using a density argument.  $\square$



### 3 Proof of Theorem 1.1

#### 3.1 Concentration near $X_\beta(\mathbf{s}_0)$

##### 3.1.1 Concentration near $X(\mathbf{s}_0)$

We shall first prove that  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) = O(\lambda^{-\infty})$ , unless  $y$  belongs to a small tubular neighborhood of  $X(\mathbf{s}_0)$ ; this will allow us to represent  $\Pi$  as an FIO with complex phase, without changing the asymptotics.

We have by (17) and (33):

$$\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) = \int_{\mathbb{R}^r} \chi_{\mathbf{s}_0}(\mathbf{s}) e^{-i\lambda \langle \beta, \mathbf{s} \rangle} \Pi(y_{\mathbf{s}}, y) d\mathbf{s}. \quad (41)$$

where  $y_{\mathbf{s}} = \phi_{-\mathbf{s}}^X(y)$ .

On the support of  $\chi_{\mathbf{s}_0}$ ,  $\|\mathbf{s} - \mathbf{s}_0\| < \epsilon$ . Hence for some  $C_1 > 0$  we have uniformly in  $y \in X$ :

$$\text{dist}(y_{\mathbf{s}_0}, y) \leq C_1 \epsilon. \quad (42)$$

On the other hand, there exist constants  $C_3 \geq C_2 > 0$  such that for every  $\epsilon > 0$  one has

$$C_3 \text{dist}_X(y, X(\mathbf{s}_0)) \geq \text{dist}(y_{\mathbf{s}_0}, y) \geq C_2 \text{dist}_X(y, X(\mathbf{s}_0)). \quad (43)$$

Indeed, since  $\phi^X$  is an action by isometries, at any  $x \in X(\mathbf{s}_0)$  we have

$$T_x(X(\mathbf{s}_0)) = \ker(d_x \phi_{\mathbf{s}_0}^X - id_{T_x X});$$

hence, there exist constants  $C'_3 \geq C'_2 > 0$  such that

$$C'_3 \|\mathbf{n}\| \geq \|d_x \phi_{\mathbf{s}_0}^X(\mathbf{n}) - \mathbf{n}\| \geq C'_2 \|\mathbf{n}\|,$$

whenever  $\mathbf{n} \in T_x(X(\mathbf{s}_0))^\perp \subseteq T_x X$ . Then (43) follows by writing, in a tubular neighborhood of  $X(\mathbf{s}_0)$ ,  $y = x + \mathbf{n}$  in a smoothly varying HLC system centered at  $x$ , and letting, say,  $C_3 = 2C'_3$ ,  $C_2 = C'_2/2$ .

Let then be  $Z_1 \subseteq X$  be the locus where  $\text{dist}_X(y, X(\mathbf{s}_0)) \geq 2(C_1/C_2)\epsilon$ . If  $y \in Z_1$ , then

$$\begin{aligned} \text{dist}(y_{\mathbf{s}}, y) &\geq \text{dist}(y_{\mathbf{s}_0}, y) - \text{dist}(y_{\mathbf{s}_0}, y_{\mathbf{s}}) \\ &\geq 2C_1\epsilon - C_1\epsilon \geq C_1\epsilon. \end{aligned} \quad (44)$$

As the singular support of  $\Pi(\cdot, \cdot) \in \mathcal{D}'(X \times X)$  is the diagonal in  $X \times X$ , it follows from (44) that the function

$$\gamma : (\mathbf{s}, y) \in \mathbb{R}^r \times Z_1 \mapsto \chi_{\mathbf{s}_0}(\mathbf{s}) \Pi(y_{\mathbf{s}}, y)$$

is well-defined and  $\mathcal{C}^\infty$ , and therefore its Fourier transform in  $\mathbf{s}$ ,

$$\widehat{\gamma}_y(\beta') =: \int_{\mathbb{R}^r} \chi_{\mathbf{s}_0}(\mathbf{s}) e^{-i\langle \beta', \mathbf{s} \rangle} \Pi(\phi_{-\mathbf{s}}^X(y), y) \, d\mathbf{s}$$

decreases rapidly for  $\beta' \rightarrow \infty$ , uniformly in  $y \in Z_1$ .

Setting  $\beta' = \lambda \beta$ , with  $\beta$  of unit norm, we have proved:

**Lemma 3.1.**  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) = O(\lambda^{-\infty})$  uniformly in  $y \in Z_1$ .

### 3.1.2 $\mathcal{S}_\chi$ as an oscillatory integral

By virtue of Lemma 3.1, in the following we can assume

$$\text{dist}_X(y, X(\mathbf{s}_0)) \leq 2(C_1/C_2)\epsilon,$$

whence for  $\chi_{\mathbf{s}_0}(\mathbf{s}) \neq 0$  we have

$$\begin{aligned} \text{dist}_X(y_{\mathbf{s}}, y) &\leq \text{dist}_X(y_{\mathbf{s}}, y_{\mathbf{s}_0}) + \text{dist}_X(y_{\mathbf{s}_0}, y) \\ &\leq C_1 \epsilon + 2(C_1 C_3/C_2) \epsilon = D_0 \epsilon, \end{aligned}$$

for some constant  $D_0 > 0$ . In this range, as in (34) and (35) we can represent  $\Pi$  as an FIO (any smoothing remainder term contributing negligibly to the asymptotics, as above).

Thus we can rewrite (41) as

$$\begin{aligned} \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) &\sim \int_0^{+\infty} \int_{\mathbb{R}^r} \chi_{\mathbf{s}_0}(\mathbf{s}) e^{i[t\psi(y_{\mathbf{s}}, y) - \lambda \langle \beta, \mathbf{s} \rangle]} s(t, y_{\mathbf{s}}, y) \, d\mathbf{s} \, dt \\ &= \lambda \int_0^{+\infty} \int_{\mathbb{R}^r} \chi_{\mathbf{s}_0}(\mathbf{s}) e^{i\lambda \Psi_\beta(y, t, \mathbf{s})} s(\lambda t, y_{\mathbf{s}}, y) \, d\mathbf{s} \, dt, \quad (45) \end{aligned}$$

where we have performed the change of variables  $t \mapsto \lambda t$ , and set

$$\Psi_\beta(y, t, \mathbf{s}) =: t\psi(y_{\mathbf{s}}, x) - \langle \beta, \mathbf{s} \rangle. \quad (46)$$

For  $D \gg 0$ , let  $\varrho = \varrho_D : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  be  $\mathcal{C}^\infty$ , identically equal to 1 on  $[1/D, D]$ , and supported in  $[1/(2D), 2D]$ .

**Lemma 3.2.** *If  $1 \gg \epsilon > 0$  and  $D \gg 0$ , only a rapidly decreasing contribution to the asymptotics of (45) is lost, if the integrand is multiplied by  $\varrho(t)$ .*

In particular, as far as the asymptotics are concerned, we may assume without loss that integration in  $t$  is compactly supported in  $[1/(2D), 2D]$ .

*Proof of Lemma 3.2.* Let  $A =: \max\{\|\Phi(m)\| : m \in M\}$ ,  $a =: \min\{\|\Phi(m)\| : m \in M\}$ . Then  $A \geq a > 0$ .

Suppose first that  $y \in X(\mathbf{s}_0)$ . Since  $d_{(y,y)}\psi = (\alpha_y, -\alpha_y)$  [BtSj], in view of (10) we have with  $m = m_y$

$$\partial_{\mathbf{s}}\psi(y_{\mathbf{s}}, y)|_{\mathbf{s}_0} = \Phi(m_y),$$

whence  $A \geq \|\partial_{\mathbf{s}}\psi(y_{\mathbf{s}}, y)|_{\mathbf{s}_0}\| \geq a$ .

Therefore, by continuity if  $\epsilon > 0$  is sufficiently small and  $\|\mathbf{s}' - \mathbf{s}_0\| < \epsilon$ ,  $\text{dist}_X(y, X(\mathbf{s}_0)) \leq 2(C_1/C_2)\epsilon$  as we are assuming then

$$2A \geq \|\partial_{\mathbf{s}}\psi(y_{\mathbf{s}}, y)|_{\mathbf{s}'}\| \geq a/2.$$

Consequently, in the same range we have

$$\|\partial_{\mathbf{s}}\Psi_{\beta}|_{\mathbf{s}'}\| = \|t\Phi(m_y) - \beta\| \geq \min\left\{\frac{1}{2}ta - 1, 1 - 2tA\right\};$$

Thus, if say  $t \geq 6/a$  then

$$\|\partial_{\mathbf{s}}\Psi_{\beta}|_{\mathbf{s}'}\| \geq \frac{1}{2} \left(\frac{t}{2} + \frac{3}{a}\right) a - 1 = \frac{t}{4} + \frac{1}{2}.$$

Similarly, if  $0 < t < 1/(3A)$ , then

$$\|\partial_{\mathbf{s}}\Psi_{\beta}|_{\mathbf{s}'}\| \geq 1 - 2tA \geq 1/3.$$

In either case, iterated integration by parts in  $\mathbf{s}$  (which is legitimate in view of the cut-off  $\chi_{\mathbf{s}_0}$ ), shows that the corresponding contribution to the asymptotics is  $O(\lambda^{-\infty})$ . The details are left to the reader.  $\square$

We have therefore

$$\begin{aligned} & \mathcal{S}_{\chi}(\lambda\beta, \mathbf{s}_0, y, y) \\ & \sim \lambda \int_{1/(2D)}^{2D} \int_{\mathbb{R}^r} e^{i\lambda\Psi_{\beta}(y,t,\mathbf{s})} \chi_{\mathbf{s}_0}(\mathbf{s}) \varrho(t) s(\lambda t, y_{\mathbf{s}}, y) \, d\mathbf{s} \, dt, \end{aligned} \quad (47)$$

where now integration is compactly supported in  $(t, \mathbf{s})$ .

### 3.1.3 Localization near $X_{\beta}(\mathbf{s}_0)$

We have already shown that (41) is rapidly decreasing outside a tubular neighborhood of  $X(\mathbf{s}_0)$  of radius  $D_1\epsilon$ . The following Lemma will show that in fact there is no loss in further restricting our analysis to an ‘oblate’ tubular neighborhood  $Z_{\epsilon}(\mathbf{s}_0, \beta)$  of  $X_{\beta}(\mathbf{s}_0)$ . As we shall see later, this result is instrumental to proving a considerably sharper asymptotic confinement property.

**Lemma 3.3.** *If  $D_2 \gg 0$ , (41) is rapidly decreasing uniformly for*

$$\text{dist}_X(y, X(\mathbf{s}_0)) \leq D_1 \epsilon \quad \text{and} \quad \text{dist}_X(y, X_\beta) \geq D_2 \epsilon.$$

*Proof of Lemma 3.3.* Suppose first that  $y \in X(\mathbf{s}_0)$  and  $\text{dist}_X(y, X_\beta) \geq D_2 \epsilon$  for some  $D_2 > 0$ . Then

$$\|\partial_{\mathbf{s}} \Psi_\beta(y, t, \mathbf{s}_0)\| = \|t \Phi(m_y) - \beta\| \geq C D_2 \epsilon,$$

where  $C > 0$  is as in Corollary 2.5.

Suppose now that  $\|\mathbf{s}' - \mathbf{s}_0\| < \epsilon$  (as will be the case for  $\chi_{\mathbf{s}_0}(\mathbf{s}') \neq 0$ ), and  $\text{dist}_X(y, X(\mathbf{s}_0)) \leq D_1 \epsilon$ . Pick  $x \in X(\mathbf{s}_0)$  with  $\text{dist}_X(y, x) \leq D_1 \epsilon$ . Then, for some appropriate  $A_1 > 0$  we have

$$\begin{aligned} \|\partial_{\mathbf{s}} \Psi_\beta(y, t, \mathbf{s}')\| &\geq \|\partial_{\mathbf{s}} \Psi_\beta(x, t, \mathbf{s}_0)\| - A_1 (\|\mathbf{s}' - \mathbf{s}_0\| + \text{dist}_X(y, x)) \\ &\geq [C D_2 - A_1 (1 + D_1)] \epsilon. \end{aligned}$$

Again, the claim follows by iterated integration by parts in  $d\mathbf{s}$ . □

In particular, since  $\phi^X$  is locally free on  $X_\beta$ , perhaps after passing to a smaller neighborhood we may assume without loss that it is locally free on  $Z_\epsilon(\mathbf{s}_0, \beta)$ .

## 3.2 Asymptotic concentration in $\mathbf{w}$ and $\tau_M$

### 3.2.1 A local parametrization of $Z_\epsilon(\mathbf{s}_0, \beta)$ by HLC

We have already remarked that  $X_\beta(\mathbf{s}_0)$  is an  $S^1$ -bundle over the union of some connected components of  $M_\beta(\mathbf{s}_0)$ , which is the transverse intersection of  $M(\mathbf{s}_0)$  and  $M_\beta$  (Lemma 2.4). We can then use a smoothly varying system of HLC centered at  $x \in X(\mathbf{s}_0, \beta)$  to locally parametrize points  $y \in Z_\epsilon(\mathbf{s}_0, \beta)$  as  $y = x + \mathbf{v}$ , where and  $\mathbf{v} \in N_{m_x}(M_\beta(\mathbf{s}_0))$  have norms bounded linearly in  $\epsilon$  (§2.1.5); in general, this is possible only locally along  $X_\beta(\mathbf{s}_0)$ .

In turn, by Lemma 2.4 we can uniquely decompose  $\mathbf{v}$  as an orthogonal direct sum  $\mathbf{v} = \mathbf{w} + J_m(\xi_M(m))$ , where  $\mathbf{w} \in \ker(d_m \phi_{\mathbf{s}}^M - \text{id}_{T_m M})^\perp$  and  $\xi \in \ker \Phi(m)$ . In addition, since  $\mathbf{s} \sim \mathbf{s}_0$ , we can write  $\mathbf{s} = \mathbf{s}_0 + \tau$ , where  $\tau$ , a small displacement in  $\mathbb{R}^r$ , is thought of as an element of  $\mathfrak{t}$ . Here  $\|\mathbf{w}\|, \|\xi\|, \|\tau\| \leq D \epsilon$  for some appropriate constant  $D$ . In this notation, Theorem 1.1 is equivalent to the statement that  $\mathcal{S}_X(\lambda \beta, \mathbf{s}_0, y, y)$  is rapidly decreasing as  $\lambda \rightarrow \infty$ , unless  $\max\{\|\mathbf{w}\|, \|\xi\|\} = O(\lambda^{\delta-1/2})$ .

In this section, we shall establish Theorem 1.2 ‘in the  $\mathbf{w}$ -direction’, and establish that locus where  $\|\tau_M(m)\| \geq C \lambda^{\delta-1/2}$  contributes negligibly to the asymptotics of (41).

### 3.2.2 The bound on $\mathbf{w}$ coming $\partial_t \Psi_\beta$

**Proposition 3.1.** *There exists a constant  $a > 0$  such that, perhaps after passing to a smaller  $\epsilon > 0$ , the following holds. Suppose  $x \in X_\beta(\mathbf{s}_0)$ ,*

$$y = y(x, \mathbf{w}, \xi) = x + \left( \mathbf{w} + J_m(\xi_M(m)) \right) \in Z_\epsilon(\mathbf{s}_0, \beta), \quad (48)$$

and  $\mathbf{s} = \mathbf{s}_0 + \tau$ . Then we have

$$\text{dist}_X(y_{\mathbf{s}}, y)^2 \geq \text{dist}_M(\pi(y)_{\mathbf{s}}, \pi(y))^2 \geq a (\|\mathbf{w}\|^2 + \|\tau_M(m_x)\|^2).$$

Before delving into the proof, let us introduce a piece of notation. We shall let  $R_j$  be a general  $\mathcal{C}^\infty$  function defined on some open neighborhood of the origin in a Euclidean space, and vanishing to  $j$ -th order at the origin;  $R_j$  is allowed to vary from line to line. Furthermore, let  $A$  be as in Notation 1.2 in §1.1.7.

*Proof of Proposition 3.1.* The first inequality is obvious, since the projection  $\pi : X \rightarrow M$  is a Riemannian submersion and intertwines  $\phi^X$  and  $\phi^M$ , so let us focus on the second.

Consider the system of adapted local coordinates on  $M$  centered at  $m_x$ , underlying the given HLC system on  $X$  centered at  $x$  [SZ]. Adapted local coordinates needn't be holomorphic, and induce a unitary isomorphism  $\mathbb{C}^d \cong T_{m_x}M$ . It is unnecessary but convenient to assume that they are given by geodesic coordinates centered at  $m_x$ . By construction,

$$\pi(y) = m_x + \left( \mathbf{w} + J_m(\xi_M(m_x)) \right). \quad (49)$$

Since  $\phi_{\mathbf{s}_0}^X$  is a Riemannian isometry fixing  $m_x$ ,

$$\begin{aligned} \pi(y)_{\mathbf{s}_0} &= m_x + d_x \phi_{-\mathbf{s}_0}^X \left( \mathbf{w} + J_m(\xi_M(m_x)) \right) \\ &= m_x + A \left( \mathbf{w} + J_m(\xi_M(m_x)) \right) = m_x + \left( A\mathbf{w} + J_m(\xi_M(m_x)) \right). \end{aligned}$$

where  $A = A_x$  is as above. Thus, since  $\mathbf{s} = \mathbf{s}_0 + \tau$ ,

$$\begin{aligned} \pi(y)_{\mathbf{s}} &= (\pi(y)_{\mathbf{s}_0})_\tau \\ &= m_x + \left( A\mathbf{w} + J_m(\xi_M(m_x)) - \tau_M(m_x) + \langle \tau, R_1(\mathbf{w}, \xi, \tau) \rangle \right). \end{aligned} \quad (50)$$

Now, since adapted local coordinates are isometric at the origin, perhaps after restricting the domain of definition we may assume that

$$2 \|\mathbf{v}_1 - \mathbf{v}_2\| \geq \text{dist}_M(m_x + \mathbf{v}_1, m_x + \mathbf{v}_2) \geq \frac{1}{2} \|\mathbf{v}_1 - \mathbf{v}_2\|. \quad (51)$$

Thus we see from (49), (50) and (51) that for some appropriate constant  $a > 0$

$$\begin{aligned} & \text{dist}_M(\pi(y)_s, \pi(y))^2 \\ & \geq \frac{1}{2 + O(\epsilon)} [\|A\mathbf{w} - \mathbf{w}\|^2 + \|\tau_M(m_x)\|^2] \geq a (\|\mathbf{w}\|^2 + \|\tau_M(m_x)\|^2), \end{aligned}$$

since  $A - I$  is invertible on  $\ker(A - I)^\perp$ .  $\square$

**Corollary 3.1.** *Let  $y = y(x, \mathbf{w}, \xi)$  be as in (48). Then for any fixed  $D, \delta > 0$  uniformly for  $\|\mathbf{w}\| \geq D \lambda^{\delta-1/2}$  we have*

$$\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) = O(\lambda^{-\infty}).$$

*Proof of Corollary 3.1.* In view of (46), we have

$$\partial_t \Psi_\beta(y, t, \mathbf{s}) = \psi(y_s, y).$$

Thus,  $|\partial_t \Psi_\beta(y, \mathbf{s})| \geq |\Im \psi(y_s, y)|$ . On the other hand, by Corollary 1.3 of [BtSj] for some  $D' > 0$  we have

$$\Im \psi(y_s, y) \geq D' \text{dist}_X(y_s, y)^2.$$

Therefore, given Proposition 3.1 and recalling that HLC are isometric at the origin,

$$|\partial_t \Psi_\beta(y, t, \mathbf{s})| = |\psi(y_s, y)| \geq |\Im \psi(y_s, y)| \geq b \|\mathbf{w}\|^2 \quad (52)$$

for some constant  $b > 0$ . For  $\|\mathbf{w}\| \geq D \lambda^{\delta-1/2}$ , therefore, we get

$$|\partial_t \Psi_\beta(y, t, \mathbf{s})| \geq b D \lambda^{2\delta-1}.$$

Hence iterated integration by parts in  $t$  introduces at each step a factor  $\lambda^{1-2\delta} \lambda^{-1} = \lambda^{-2\delta}$ .  $\square$

The same argument proves the following:

**Corollary 3.2.** *For any fixed  $D > 0$ , the locus  $(\tau, t)$  where  $\|\tau_M(m_x)\| \leq D \lambda^{\delta-1/2}$  contributes negligibly to the asymptotics of  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)$ .*

$$\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) = O(\lambda^{-\infty}),$$

*uniformly for  $\|\tau_M(m_x)\| \geq D \lambda^{\delta-1/2}$ .*

Thus without loss we may assume from now on that for some fixed  $D > 0$

$$\|\mathbf{w}\| \leq D \lambda^{\delta-1/2}, \quad (53)$$

and restrict our analysis of the oscillatory integral (47) to the locus

$$\{\tau \in \mathfrak{t} : \|\tau_M(m_x)\| \leq D \lambda^{\delta-1/2}\}. \quad (54)$$

More precisely, we have

**Corollary 3.3.** *Only a rapidly decreasing contribution to the asymptotics of  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)$  is lost, if the integrand is multiplied by a rescaled cut-off function of the form  $\rho(\lambda^{1/2-\delta} \tau_M(m_x))$ , where  $\rho$  is compactly supported in a neighborhood of the origin, and identically equal to 1 sufficiently close to  $\mathbf{0} \in \mathbb{R}^r$ .*

Let us set  $\mathbf{s}_\tau =: \mathbf{s}_0 + \tau$  for brevity. Then we get

$$\begin{aligned} & \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) \\ & \sim \lambda \int_{1/(2D)}^{2D} \int_{\mathfrak{t}} e^{i\lambda \Psi_\beta(y, t, \mathbf{s}_\tau)} \rho(\lambda^{1/2-\delta} \tau_M(m_x)) \chi(\tau) \varrho(t) s(\lambda t, y_{\mathbf{s}_\tau}, z) d\tau dt. \end{aligned} \quad (55)$$

However, the latter asymptotic equality does not yet allow us to reduce to the case  $\|\tau\| \leq D \lambda^{\delta-1/2}$ , because the evaluation map  $\tau \mapsto \tau_M(m_x)$  needn't be injective.

### 3.2.3 Domain concentration in $\tau$ coming from $\text{dist}_X(y_{\mathbf{s}}, y)$ and $\partial_t \Psi_\beta$

With  $\mathbf{s}_\tau = \mathbf{s}_0 + \tau$  as above, and a constant  $D > 0$ , let us set

$$B_\lambda(y) =: \{\tau \in \mathfrak{t} : \text{dist}_X(y_{\mathbf{s}_\tau}, y) \geq D \lambda^{\delta-1/2}\}.$$

The same argument used in the proof of Corollary 3.1 implies the following

**Corollary 3.4.** *The locus  $B_\lambda(y)$  contributes negligibly to the asymptotics of  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)$*

Using first Lemma 3.2 of [P5] and then Corollary 2.2 of [P4], with the

given choice of geodesic adapted coordinates centered at  $m_x$  we get

$$\begin{aligned}
y_{\mathbf{s}_\tau} &= \phi_{-\tau-\mathbf{s}_0}^X \left( x + (\mathbf{w} + J_m(\xi_M(m_x))) \right) \\
&= \phi_{-\tau}^X \circ \phi_{-\mathbf{s}_0}^X \left( x + (\mathbf{w} + J_{m_x}(\xi_M(m_x))) \right) \\
&= \phi_{-\tau}^X \left( x + \left( R_3(\mathbf{w}, \xi), A\mathbf{w} + J_m(\xi_M(m_x)) \right) \right) \\
&= x + \left( \langle \Phi(m_x), \tau \rangle + \omega_{m_x} \left( \tau_M(m_x), A\mathbf{w} + J_{m_x}(\xi_M(m_x)) \right) + R_3(\tau, \mathbf{w}, \xi), \right. \\
&\quad \left. A\mathbf{w} + J_{m_x}(\xi_M(m_x)) - \tau_M(m_x) + R_2(\tau, \mathbf{w}, \xi) \right) \\
&= x + \left( \langle \Phi(m_x), \tau \rangle + g_{m_x}(\tau_M(m_x), \xi_M(m_x)) + R_3(\tau, \mathbf{w}, \xi), \right. \\
&\quad \left. A\mathbf{w} + J_m(\xi_M(m_x)) - \tau_M(m_x) + R_2(\tau, \mathbf{w}, \xi) \right), \tag{56}
\end{aligned}$$

where  $A$  is again as in §1.1.7. We have used that  $\tau_M(m_x)$  and  $A\mathbf{w}$  live in orthogonal complex subspaces (with respect to the Hermitian structure of  $T_{m_x}M$ ), and therefore are symplectically orthogonal as well.

Since HLC are isometric at the origin, perhaps after restricting the domain of definition we have

$$\begin{aligned}
2 \left\| (\theta_1 - \theta_2, \mathbf{v}_1 - \mathbf{v}_2) \right\| &\geq \text{dist}_X(x + (\theta_1, \mathbf{v}_1), x + (\theta_2, \mathbf{v}_2)) \\
&\geq \frac{1}{2} \left\| (\theta_1 - \theta_2, \mathbf{v}_1 - \mathbf{v}_2) \right\| = \frac{1}{2} \sqrt{(\theta_1 - \theta_2)^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|^2}. \tag{57}
\end{aligned}$$

Thus we see from (56) that, with  $y$  as in (48),

$$\begin{aligned}
&\text{dist}_X(y_{\mathbf{s}_\tau}, y) \tag{58} \\
&\geq \frac{1}{2} \left| \langle \Phi(m_x), \tau \rangle + g_{m_x}(\tau_M(m_x), \xi_M(m_x)) \right| + R_3(\tau, \mathbf{w}, \xi).
\end{aligned}$$

By virtue of Corollary 3.4, we obtain the following:

**Lemma 3.4.** *Let  $y = y(x, \mathbf{w}, \xi)$  be as in (48). Given a constant  $E > 0$ , the locus of those  $\tau \in \mathfrak{t}$  such that*

$$\left| \langle \Phi(m_x), \tau \rangle + g_{m_x}(\tau_M(m_x), \xi_M(m_x)) \right| + R_3(\tau, \mathbf{w}, \xi) \geq E \lambda^{\delta-1/2},$$

*contributes negligibly to the asymptotics of  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)$ .*

We conclude that there is no loss of generality in further restricting integration in  $d\tau$  in (55) to the locus in  $\mathfrak{t}$  where

$$\left| \langle \Phi(m_x), \tau \rangle + g_{m_x}(\tau_M(m_x), \xi_M(m_x)) \right| + R_3(\tau, \mathbf{w}, \xi) < E \lambda^{\delta-1/2}. \tag{59}$$

This may be accomplished  $\mathcal{C}^\infty$ -wise by redefining  $\rho$  if necessary (but with same type of scaling), and will be left implicit in the following.



### 3.2.4 Domain concentration coming from $\partial_\tau \psi_\beta$

Since  $\Psi_\beta$  is complex valued,  $\partial_\tau \Psi_\beta(y, t, \mathbf{s}_\tau) \in \mathfrak{t}^\vee \otimes \mathbb{C}$  (recall that  $\mathbf{s}_\tau = \mathbf{s}_0 + \tau$ ). Let us now define

$$\begin{aligned} A'_\lambda(y) &=: \{(t, \tau) : \|\partial_\tau \Psi_\beta(y, t, \mathbf{s}_\tau)\| < 2D \lambda^{\delta-1/2}\}, \\ A''_\lambda(y) &=: \{(t, \tau) : \|\partial_\tau \Psi_\beta(y, t, \mathbf{s}_\tau)\| > D \lambda^{\delta-1/2}\}. \end{aligned} \quad (60)$$

Then  $\{A'_\lambda(y, \tau), A''_\lambda(y, \tau)\}$  is an open cover of  $\mathfrak{t}$ , and we may find a partition of unity subordinate to it,  $\{\varsigma_\lambda, 1 - \varsigma_\lambda\}$  of the form

$$\varsigma_\lambda(t, \tau) =: \varsigma(\lambda^{1/2-\delta} \partial_\tau \Psi_\beta(y, t, \mathbf{s}_\tau)),$$

for an appropriate bump function  $\varsigma \in \mathcal{C}_0^\infty(\mathfrak{t}^\vee \otimes \mathbb{C})$ , supported in an open ball of radius  $2D$  centered at the origin  $\mathbf{0} \in \mathfrak{t}$ , and identically equal to 1 within distance  $D$  from the origin. We can then rewrite (55) as

$$\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) = \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)' + \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)'', \quad (61)$$

where  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)'$  and  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)''$  are given by (55), but with the integrand multiplied by  $\varsigma_\lambda(\tau)$  and  $1 - \varsigma_\lambda(\tau)$ , respectively.

**Proposition 3.2.**  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)'' = O(\lambda^{-\infty})$  for  $\lambda \rightarrow \infty$ .

*Proof of Proposition 3.2.* Let us define

$$Z(\partial_\tau \Psi_\beta, y) =: \{\tau \in \mathfrak{t} : \partial_\tau \Psi_\beta(y, \mathbf{s}_\tau) = \mathbf{0}\}.$$

Let  $(X_j)$  be the standard linear coordinates on  $\mathfrak{t} \cong \mathbb{R}^r$ . On  $\mathfrak{t} \setminus Z(\partial_\tau \Psi_\beta, y)$ , we may consider the differential operator

$$L =: \frac{1}{\sum_{j=1}^r |\partial_{X_j} \Psi_\beta(y, \mathbf{s}_\tau)|^2} \sum_j \partial_{X_j} \overline{\Psi_\beta(y, \mathbf{s}_\tau)} \partial_{X_j}.$$

Then  $L(\Psi_\beta) = 1$ , and so  $L(e^{i\lambda \Psi_\beta}) = i\lambda e^{i\lambda \Psi_\beta}$ . Let us also define

$$\rho_\lambda(\tau) =: \rho(\lambda^{1/2-\delta} \tau_M(m_x)) (1 - \varsigma_\lambda(\tau)) \quad (\tau \in \mathfrak{t}),$$

$$\mathcal{A}_\lambda(y, \tau, t) =: \rho_\lambda(\tau) \varrho(t) \chi(\tau) s(\lambda t, y_{\mathbf{s}_\tau}, y).$$

Then we obtain

$$\begin{aligned}
& \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)'' \tag{62} \\
& \sim \lambda \int_{1/(2D)}^{2D} \int_{\mathbf{t}} e^{i\lambda \Psi_\beta(y, t, \mathbf{s}_\tau)} \mathcal{A}_\lambda(y, \tau, t) d\tau dt \\
& = -i \int_{1/(2D)}^{2D} \int_{\mathbf{t}} L(e^{i\lambda \Psi_\beta(y, t, \mathbf{s}_\tau)}) \mathcal{A}_\lambda(y, \tau, t) d\tau dt \\
& = i \sum_j \int_{1/(2D)}^{2D} \int_{\mathbf{t}} e^{i\lambda \Psi_\beta(y, t, \mathbf{s}_\tau)} \partial_{X_j} \left( \frac{\overline{\partial_{X_j} \Psi_\beta(y, \mathbf{s}_\tau)}}{\sum_l |\partial_{X_l} \Psi_\beta(y, \mathbf{s}_\tau)|^2} \mathcal{A}_\lambda(y, \tau, t) \right) d\tau dt \\
& = -i \sum_j \int_{1/(2D)}^{2D} \int_{\mathbf{t}} e^{i\lambda \Psi_\beta(y, t, \mathbf{s}_\tau)} P(\mathcal{A}_\lambda(y, \tau, t)) d\tau dt,
\end{aligned}$$

where  $P = L^\dagger$  is the transpose operator, given by

$$P(h) =: - \sum_{j=1}^r \partial_{X_j} \left( \frac{\partial_{X_j} \overline{\Psi_\beta(y, \mathbf{s}_\tau)}}{\sum_l |\partial_{X_l} \Psi_\beta(y, \mathbf{s}_\tau)|^2} \cdot h \right).$$

Using the asymptotic expansion of  $s(\lambda t, z_{\mathbf{s}_0+\tau}, z)$ , one sees that the integrand on the last line of (62) is bounded by  $C_k \lambda^{d+1-2\delta}$ .

Iterating the integration by parts in  $\tau$ , as above, we obtain for any  $k \geq 1$

$$\begin{aligned}
& \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)'' \tag{63} \\
& \sim (-i)^k \lambda^{1-k} \int_{1/(2D)}^{2D} \int_{\mathbf{t}} L^k(e^{i\lambda \Psi_\beta(y, t, \mathbf{s}_\tau)}) \mathcal{A}_\lambda(y, \tau, t) d\tau dt \\
& = (-i)^k \lambda^{1-k} \int_{1/(2D)}^{2D} \int_{\mathbf{t}} e^{i\lambda \Psi_\beta(y, t, \mathbf{s}_\tau)} P^k(\mathcal{A}_\lambda(y, \tau, t)) d\tau dt,
\end{aligned}$$

One can then check inductively the following:

**Lemma 3.5.** *Let us set  $V_j =: \partial_{X_j} \Psi_\beta(y, \mathbf{s}_\tau)$ , and  $V = (V_j)$ . Then, for any  $k \geq 1$ ,  $P^k(\mathcal{A}_\lambda(\tau, t))$  is a linear combination of terms of the form*

$$\frac{\mathcal{P}_a(V, \overline{V})}{\|V\|^{2b}} \lambda^{c(1/2-\delta)} B_\lambda(\tau, t),$$

where  $\mathcal{P}_a$  is a homogeneous polynomial of degree  $a$ , with coefficients depending on the derivatives of  $V$ , and  $a, b, c \in \mathbb{N}$ ,  $2b - a + c \leq 2k$ ; also,  $|B_\lambda| \leq C'_{a,b,c} \lambda^d$  for  $\lambda \gg 0$ .

The bound on  $B_\lambda$  follows from the asymptotic expansion for the amplitude  $s$  of  $\Pi$  in (34).

On the other hand, in view of the definition of  $A''_\lambda(y)$  in (60), on the support of  $1 - \varsigma_\lambda(\tau)$  each summand in Lemma 3.5 satisfies an estimate of the form

$$\begin{aligned} \left| \frac{\mathcal{P}_a(V)}{\|V\|^{2b}} \lambda^{c(1/2-\delta)} B_\lambda(\tau, t) \right| &\leq C_{a,b,c} \frac{\|V\|^a}{\|V\|^{2b}} \lambda^{d+c(1/2-\delta)} \\ &= C_{a,b,c} \frac{1}{\|V\|^{2b-a}} \lambda^{d+c(1/2-\delta)} \leq D_{a,b,c} \lambda^{d+k(1-2\delta)} \end{aligned}$$

as  $\lambda \rightarrow +\infty$ . Inserting this in (63), we obtain an upper bound of the form  $C_k \lambda^{d+1-2k\delta}$ . This completes the proof of the Proposition.  $\square$

We conclude from (61) and Proposition 3.2 that

$$\begin{aligned} \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) &\sim \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y)' \\ &\sim \lambda \int_{1/(2D)}^{2D} \int_{\mathbf{t}} e^{i\lambda \Psi_\beta(y, t, \mathbf{s}_\tau)} \mathcal{B}_\lambda(y, \tau, t) \, d\tau \, dt \end{aligned} \tag{64}$$

where now

$$\mathcal{B}_\lambda(y, \tau, t) =: \rho \left( \lambda^{1/2-\delta} \tau_M(m_x) \right) \varsigma_\lambda(t, \tau) \varrho(t) \chi(\tau) s(\lambda t, y_{\mathbf{s}_\tau}, y). \tag{65}$$

The domain of integration in (64) is then  $A'_\lambda(y)$ .

### 3.2.5 The reduction in $\tau$ and the bound in $\xi$

We shall now combine (54), (59) and the domain reduction obtained in (64), always assuming (53).

Since  $\tau \mapsto \tau_X(x)$  is injective whenever  $x \in X_\beta$ , we have  $\|\tau_X(x)\| \geq a \|\tau\|$  for some constant  $a = a_\beta > 0$ , depending only on  $\beta$ .

On the other hand, in HLC centered at  $x$  we have

$$\|\tau_X(x)\| = \left\| \left( \langle \Phi(m_x), \tau \rangle, -\tau_M(m_x) \right) \right\|.$$

On the domain of integration of (55), we are in the range (54); therefore,

$$|\langle \Phi(m_x), \tau \rangle| \geq a \|\tau\| + O(\lambda^{\delta-1/2}). \tag{66}$$

On the other hand, we are now assuming that on the same domain (59) also holds; combining (66) with (59), we conclude that on the domain of

integration of (55), further reduced according to (59), we have for appropriate constants  $D_1, D'_1 > 0$ :

$$\begin{aligned} E \lambda^{\delta-1/2} &\geq \left| \langle \Phi(m_x), \tau \rangle + g_m(\tau_M(m_x), \xi_M(m_x)) + R_3(\tau, \mathbf{w}, \xi) \right| \\ &\geq D_1 \|\tau\| + R_3(\tau, \mathbf{w}, \xi) + O(\lambda^{\delta-1/2}) \\ &\geq D'_1 \|\tau\| + R_3(\xi) + O(\lambda^{\delta-1/2}). \end{aligned}$$

We have used that  $\|\tau\| \gg R_j(\tau)$  for  $j = 2, 3$  e  $\|\tau\| < \epsilon$ ,  $\epsilon$  small. Therefore we obtain the following:

**Lemma 3.6.** *In the domain of integration of (55), and with the reduction (59) implicit, for some constant  $D_5 > 0$  we have*

$$\|\tau\| \leq D_5 \|\xi\|^3 + O(\lambda^{\delta-1/2}).$$

On the other hand, in view of (51), given (53) and (54) we have

$$\text{dist}_M(\pi(y)_{\mathbf{s}}, \pi(y)) \leq 4D \lambda^{\delta-1/2}.$$

Given that  $\pi : X \rightarrow M$  is a Riemannian submersion with fibers the  $S^1$ -orbits in  $X$ , there exists  $\vartheta = \vartheta(y, \mathbf{s}) \in (-\pi, \pi]$  such that

$$\text{dist}_X(y_{\mathbf{s}}, e^{i\vartheta} y) \leq 4D \lambda^{\delta-1/2}.$$

In view of Remark 2.1, identifying  $d\psi$  with its local coordinate expression,

$$\begin{aligned} d_{(y_{\mathbf{s}}, y)} \psi &= d_{(e^{i\vartheta} y, y)} \psi + O(\lambda^{\delta-1/2}) \\ &= (e^{i\vartheta} \alpha_{e^{i\vartheta} y}, -e^{i\vartheta} \alpha_y) + O(\lambda^{\delta-1/2}) \\ &= (e^{i\vartheta} \alpha_y, -e^{i\vartheta} \alpha_y) + O(\lambda^{\delta-1/2}), \end{aligned} \tag{67}$$

where on the last line we have used that  $\alpha$  is  $S^1$ -invariant, and therefore it does not depend on the  $\theta$ -coordinate in a HLC system (recall that in HLC the  $S^1$ -action on  $X$  is expressed by a translation in the angular coordinate).

Given  $\xi \in \ker \Phi(m)$ , let us introduce the linear functional on  $\mathfrak{t}$

$$L_m(\xi) : \tau \mapsto g_m(\tau_M(m), \xi_M(m)).$$

Given (67) and (56), we see from (46) that

$$\begin{aligned} \partial_\tau \Psi_\beta(y, t, \mathbf{s}_\tau) &= e^{i\vartheta} \left( t \Phi(m_x) + L_m(\xi) \right) - \beta + R_2(\tau, \xi) + O(\lambda^{\delta-1/2}) \\ &= [e^{i\vartheta} t \Phi(m_x) - \beta] + e^{i\vartheta} L_m(\xi) + R_2(\tau, \xi) + O(\lambda^{\delta-1/2}). \end{aligned}$$

**Lemma 3.7.** *In the range of the present discussion,*

$$\|\partial_\tau \Psi_\beta(y, t, \mathbf{s}_\tau)\| \geq b \|\xi\| + R_2(\tau) + O(\lambda^{\delta-1/2})$$

for some constant  $b > 0$ .

*Proof of Lemma 3.7.* We have

$$e^{i\vartheta} t \Phi(m) - \beta = e^{i\vartheta} t \Phi(m) - \Phi_u(m) \in \text{span}_{\mathbb{C}}\{\Phi(m)\} \subseteq \mathfrak{t}^\vee \otimes \mathbb{C},$$

while every non-zero element of

$$\mathcal{L}_m =: \left\{ L_m(\xi) : \xi \in \ker \Phi(m) \right\} \otimes \mathbb{C} \subseteq \mathfrak{t}^\vee \otimes \mathbb{C}$$

is non-vanishing on  $\ker \Phi(m)$ , as the evaluation map  $\text{val}_m : \mathfrak{t} \rightarrow T_m M$  is injective on  $\ker \Phi(m)$ ; in particular  $L_m(\xi)(\xi) = \|\xi_M(m)\|^2 > 0$  for any  $\xi \in \ker \Phi(m)$ ,  $\xi \neq 0$ . Hence  $\xi \in \ker \Phi(m) \mapsto L_m(\xi) \in \mathcal{L}_m$  is an isomorphism, and

$$\mathcal{L}_m \cap \text{span}\{\Phi(m)\} = (\mathbf{0});$$

this implies for some constants  $a_1, a_2 > 0$  and every  $\xi, t$

$$\left\| L_m(\xi) + \left( e^{i\vartheta} t \Phi(m) - \beta \right) \right\| \geq a_1 \left( \|L_m(\xi)\| + \left\| e^{i\vartheta} t \Phi(m) - \beta \right\| \right) \geq a_2 \|\xi\|.$$

To complete the proof, we need only remark that  $\|\xi\| \gg R_2(\xi), R_1(\tau), R_1(\xi)$ , since  $\tau$  and  $\xi$  are bounded linearly in  $\epsilon$ , and  $\epsilon$  is assumed very small.  $\square$

On the domain of integration  $A'_\lambda(y)$ , we then obtain

$$\|\xi\| + R_2(\tau) = O(\lambda^{\delta-1/2}) \implies \|\xi\| \leq A \|\tau\|^2 + O(\lambda^{\delta-1/2}). \quad (68)$$

Pairing (68) with the bound in Lemma 3.6, we obtain first that that in the domain of integration of (64) we have

$$\|\tau\| \leq C' \lambda^{\delta-1/2}, \quad \|\xi\| \leq C'' \lambda^{\delta-1/2} \quad (69)$$

for appropriate constants  $C', C'' > 0$ .

### 3.2.6 Proof of Theorem 1.1

Summing up, we have established that for every  $\delta \in (0, 1/2)$  and any given positive constant  $a_\delta > 0$  there exists  $b_\delta > 0$  such that  $\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) = O(\lambda^{-\infty})$  with  $y = y(x, \mathbf{w}, \xi)$  as in (48), if  $\|\mathbf{w}\| \geq a_\delta \lambda^{\delta-1/2}$  or  $\|\xi\| \geq b_\delta \lambda^{\delta-1/2}$  for  $\lambda \gg 0$ . Let us now choose an arbitrary constant  $a > 0$  and suppose that  $\max\{\|\mathbf{w}\|, \|\xi\|\} \geq a \lambda^{\delta-1/2}$ . Choose  $\delta' \in (0, \delta)$ . Then for  $\lambda \gg 0$  we have

$$\max\{\|\mathbf{w}\|, \|\xi\|\} \geq a \lambda^{\delta-1/2} > \max\{a_{\delta'}, b_{\delta'}\} \lambda^{\delta'-1/2}.$$

We conclude the following:

**Corollary 3.5.** *For any positive constant  $a > 0$ , we have  $\mathcal{S}_\chi(\lambda\beta, \mathbf{s}_0, y, y) = O(\lambda^{-\infty})$  with  $y = y(x, \mathbf{w}, \xi)$ , uniformly in  $(\mathbf{w}, \xi)$  satisfying  $\max\{\|\mathbf{w}\|, \|\xi\|\} \geq a\lambda^{\delta-1/2}$  for  $\lambda \gg 0$ .*

*Proof of Theorem 1.1.* If (21) holds with  $y = y(x, \mathbf{w}, \xi)$ , then in view of (57) we need to have

$$\max\{\|\mathbf{w}\|, \|\xi\|\} \geq \frac{1}{\sqrt{2}} \sqrt{\|\mathbf{w}\|^2 + \|\xi\|^2} \geq \frac{1}{2\sqrt{2}} \text{dist}_X(y, x) \geq \frac{D}{2\sqrt{2}} \lambda^{\delta-1/2}.$$

Thus the statement of the Theorem follows from Corollary 3.5.  $\square$

## 4 Proof of Theorem 1.2 and Corollary 1.2

Before delving into the proof, let us note that in the course of the proof of Theorem 1.1 we have established the following: in (64) only a negligible contribution to the asymptotics is lost, if integration in  $\tau$  is restricted to a neighborhood of origin of radius  $C'\lambda^{\delta-1/2}$ , for some  $C' > 0$  (see (69)). Hence we may rewrite (64) as follows:

$$\mathcal{S}_\chi(\lambda\beta, \mathbf{s}_0, y, y) \sim \lambda \int_{1/(2D)}^{2D} \int_{\mathbf{t}} e^{i\lambda \Psi_\beta(y, t, \mathbf{s}_\tau)} \mathcal{C}_\lambda(y, \tau, t) d\tau dt$$

where now

$$\mathcal{C}_\lambda(y, \tau, t) =: \gamma(\lambda^{1/2-\delta}\tau) \varrho(t) \chi(\tau) s(\lambda t, y_{\mathbf{s}_\tau}, y), \quad (70)$$

with  $\gamma \in \mathcal{C}_0^\infty(\mathbf{t})$  compactly supported and identically equal to one on an appropriate neighborhood of the origin. In view of (46), we can further rewrite (70) as follows:

$$\begin{aligned} \mathcal{S}_\chi(\lambda\beta, \mathbf{s}_0, y, y) \\ \sim \lambda e^{-i\lambda \langle \beta, \mathbf{s}_0 \rangle} \int_{1/(2D)}^{2D} \int_{\mathbf{t}} e^{i\lambda \Upsilon_\beta(y, t, \tau)} \mathcal{C}_\lambda(y, \tau, t) d\tau dt, \end{aligned} \quad (71)$$

where

$$\Upsilon_\beta(y, t, \tau) =: t\psi(y_{\mathbf{s}_\tau}, y) - \langle \beta, \tau \rangle; \quad (72)$$

here  $y_{\mathbf{s}_\tau}$  is given by (56).

*Proof of Theorem 1.2.* Let us set, in the notation of §1.1.7 and with  $\mathbf{v}$  as in (23),

$$\mathbf{s}_\lambda =: \mathbf{s}_0 + \frac{1}{\sqrt{\lambda}} \tau \in \mathbb{R}^r, \quad y_\lambda = x + \frac{1}{\sqrt{\lambda}} \mathbf{v} \in X, \quad y_{\lambda, \mathbf{s}_\lambda} =: \phi_{-\mathbf{s}_\lambda}^X(y_\lambda) \in X, \quad (73)$$

$$m_\lambda =: m_x + \frac{1}{\sqrt{\lambda}} \mathbf{v} = \pi(y_\lambda) \in M, \quad m_{\lambda, \mathbf{s}_\lambda} =: \phi_{-\mathbf{s}_\lambda}^M(m_\lambda) = \pi(y_{\lambda, \mathbf{s}_\lambda}) \in M. \quad (74)$$

With the change of integration variable  $\tau \mapsto \tau/\sqrt{\lambda}$ , (71) may be rewritten

$$\begin{aligned} \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y_\lambda, y_\lambda) & \quad (75) \\ \sim \lambda^{1-\frac{r}{2}} e^{-i\lambda \langle \beta, \mathbf{s}_0 \rangle} \int_{1/(2D)}^{2D} \int_{\mathbf{t}} e^{i\lambda \Upsilon_\beta(y_\lambda, t, \tau/\sqrt{\lambda})} \mathcal{D}_\lambda(y, \tau, t) \, d\tau \, dt, \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}_\lambda(y, \tau, t) & =: \mathcal{C}_\lambda \left( y_\lambda, \frac{1}{\sqrt{\lambda}} \tau, t \right) \\ & = \gamma(\lambda^{-\delta} \tau) \varrho(t) \chi(\lambda^{-1/2} \tau) s(\lambda t, y_{\lambda, \mathbf{s}_\lambda}, y_\lambda), \end{aligned} \quad (76)$$

and integration in  $d\tau$  is now over an expanding ball in  $\mathbf{t}$  centered at the origin and radius  $O(\lambda^\delta)$ .

Let us compute the phase in (75). We have by (72)

$$\Upsilon_\beta \left( y_\lambda, t, \frac{\tau}{\sqrt{\lambda}} \right) = t \psi(y_{\lambda, \mathbf{s}_\lambda}, y_\lambda) - \frac{1}{\sqrt{\lambda}} \langle \beta, \tau \rangle. \quad (77)$$

Now let us apply (56) with  $\tau$ ,  $\mathbf{w}$ , and  $\xi$  rescaled by  $1/\sqrt{\lambda}$ . We obtain

$$\begin{aligned} y_{\lambda, \mathbf{s}_\lambda} & = \phi_{-\frac{\tau}{\sqrt{\lambda}} - \mathbf{s}_0}^X(y_\lambda) \quad (78) \\ & = x + \left( \frac{1}{\sqrt{\lambda}} \langle \Phi(m_x), \tau \rangle + \frac{1}{\lambda} g_{m_x}(\tau_M(m_x), \xi_M(m_x)) + R_3 \left( \frac{\tau}{\sqrt{\lambda}}, \frac{\mathbf{w}}{\sqrt{\lambda}}, \frac{\xi}{\sqrt{\lambda}} \right), \right. \\ & \quad \left. \frac{1}{\sqrt{\lambda}} (A\mathbf{w} + J_{m_x}(\xi_M(m_x)) - \tau_M(m_x)) + R_2 \left( \frac{\tau}{\sqrt{\lambda}}, \frac{\mathbf{w}}{\sqrt{\lambda}}, \frac{\xi}{\sqrt{\lambda}} \right) \right) \\ & = x + (\Theta_\lambda, V_\lambda), \end{aligned}$$

where  $\Theta_\lambda = \Theta_\lambda(x, \mathbf{w}, \xi, \tau)$  and  $V_\lambda = V_\lambda(x, \mathbf{w}, \xi, \tau)$  are defined by the previous

equality. In view of Remark 2.2, (78) implies

$$\begin{aligned}
t \psi(y_{\lambda, s_\lambda}, y_\lambda) &= t \psi \left( x + (\Theta_\lambda, V_\lambda), x + \frac{1}{\sqrt{\lambda}} [\mathbf{w} + J_{m_x}(\xi_M(m_x))] \right) \\
&= it [1 - e^{i\Theta_\lambda}] - it \psi_2 \left( V_\lambda, \frac{1}{\sqrt{\lambda}} [\mathbf{w} + J_{m_x}(\xi_M(m_x))] \right) e^{i\Theta_\lambda} \\
&\quad + t R_3 \left( V_\lambda, \frac{1}{\sqrt{\lambda}} [\mathbf{w} + J_{m_x}(\xi_M(m_x))] \right) e^{i\Theta_\lambda} \\
&= it [1 - e^{i\Theta_\lambda}] - \frac{it}{\lambda} \psi_2 \left( A\mathbf{w} + J_m(\xi_M(m_x)) - \tau_M(m_x), \mathbf{w} + J_{m_x}(\xi_M(m_x)) \right) e^{i\Theta} \\
&\quad + t R_3 \left( \frac{\tau}{\sqrt{\lambda}}, \frac{\mathbf{w}}{\sqrt{\lambda}}, \frac{\xi}{\sqrt{\lambda}} \right) e^{i\Theta_\lambda}
\end{aligned} \tag{79}$$

(recall that  $R_3$  is a generic  $\mathcal{C}^\infty$  function vanishing to third order at the origin, and is allowed to vary from line to line). Inserting (79) in (77), we get with some computations

$$\begin{aligned}
i\lambda \Upsilon_\beta \left( y_\lambda, t, \frac{\tau}{\sqrt{\lambda}} \right) & \\
&= i\sqrt{\lambda} \langle t\Phi(m_x) - \beta, \tau \rangle + t \left[ i g_{m_x}(\xi_M(m_x), \tau_M(m_x)) - \frac{1}{2} \langle \Phi(m_x), \tau \rangle^2 \right. \\
&\quad \left. + \psi_2 \left( J(\xi_M(m_x)) + A\mathbf{w} - \tau_M(m_x), J(\xi_M(m_x)) + \mathbf{w} \right) \right] \\
&\quad + \lambda t R_3 \left( \frac{\tau}{\sqrt{\lambda}}, \frac{\mathbf{w}}{\sqrt{\lambda}}, \frac{\xi}{\sqrt{\lambda}} \right).
\end{aligned} \tag{80}$$

We have

$$\begin{aligned}
&\psi_2 \left( J(\xi_M(m_x)) + A\mathbf{w} - \tau_M(m_x), J(\xi_M(m_x)) + \mathbf{w} \right) \\
&= \psi_2(A\mathbf{w}, \mathbf{w}) + i g_{m_x}(\xi_M(m_x), \tau_M(m_x)) - \frac{1}{2} \|\tau_M(m_x)\|^2.
\end{aligned}$$

In particular, on the domain of integration and in the range of the Theorem we have for some  $c > 0$

$$\begin{aligned}
\Re \left( i\lambda \Upsilon_\beta \left( z_\lambda, t, \tau/\sqrt{\lambda} \right) \right) &\leq -\frac{t}{2} \|A\mathbf{w} - \mathbf{w}\|^2 \\
&\quad - \frac{t}{2} \left( \langle \Phi(m_x), \tau \rangle^2 + \|\tau_M(m_x)\|^2 \right) + \Re \left( \lambda R_3 \left( \frac{\tau}{\sqrt{\lambda}}, \frac{\mathbf{w}}{\sqrt{\lambda}}, \frac{\xi}{\sqrt{\lambda}} \right) \right) \\
&\leq -c (\|\mathbf{w}\|^2 + \|\tau\|^2) + O(\lambda^{3\delta-1/2}) = -c (\|\mathbf{w}\|^2 + \|\tau\|^2) + o(1)
\end{aligned}$$

since by assumption  $0 < \delta < 1/6$ .



Let  $\Xi(m) \in \mathfrak{t}$  be as in §2.1.2, so that we have an orthogonal direct sum

$$\mathfrak{t} = \text{span}(\Xi(m_x)) \oplus \ker(\Phi(m_x)). \quad (81)$$

For any  $\tau \in \mathfrak{t}$ , we can write, for unique  $u \in \mathbb{R}$  and  $\eta \in \ker \Phi(m_x)$ ,

$$\tau = \tau(u, \eta) =: u \Xi(m_x) + \eta. \quad (82)$$

Recalling that  $\beta = \Phi_u(m) = \Phi(m)/\|\Phi(m)\|$ , (80) may be rewritten as follows:

$$\begin{aligned} & i\lambda \Upsilon_\beta \left( z_\lambda, t, \tau/\sqrt{\lambda} \right) \\ &= i\sqrt{\lambda} \Gamma_x(t, u) + t \mathcal{E}_x(\xi, \tau(u, \eta), \mathbf{w}) + \lambda t R_3 \left( \frac{\tau}{\sqrt{\lambda}}, \frac{\mathbf{w}}{\sqrt{\lambda}}, \frac{\xi}{\sqrt{\lambda}} \right), \end{aligned} \quad (83)$$

where

$$\begin{aligned} \Gamma_x(t, u) &=: (\|\Phi(m_x)\| t - 1) u, \\ \mathcal{E}_x(\xi, \tau(u, \eta), \mathbf{w}) &=: \psi_2(A \mathbf{w}, \mathbf{w}) - \frac{1}{2} \|\Phi(m_x)\|^2 u^2 \\ &\quad + 2i g_{m_x}(\xi_M(m_x), \tau(u, \eta)_M(m_x)) - \frac{1}{2} \|\tau(u, \eta)_M(m_x)\|^2. \end{aligned} \quad (84)$$

In view of (81) and (82), we can write the integral over  $\mathfrak{t}$  as an iterated integral:

$$\int_{\mathfrak{t}} d\tau = \int_{\ker \Phi(m_x)} d\eta \int_{-\infty}^{+\infty} du.$$

We shall then rewrite (75) in the following form:

$$\mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y_\lambda, y_\lambda) \sim \lambda^{1-\frac{r}{2}} e^{-i\lambda \langle \beta, \mathbf{s}_0 \rangle} \int_{\ker \Phi(m_x)} I_\lambda(x, \eta, \mathbf{w}, \xi) d\eta, \quad (85)$$

where the inner integral

$$\begin{aligned} & I_\lambda(x, \eta, \mathbf{w}, \xi) \\ &=: \int_{1/(2D)}^{2D} \int_{-\infty}^{+\infty} e^{i\sqrt{\lambda} \Gamma(t, u)} e^{t \mathcal{E}} e^{t \lambda R_3} \cdot \mathcal{D}_\lambda(y, \tau, t) du dt \end{aligned} \quad (86)$$

is an oscillatory integral in  $\sqrt{\lambda}$ , with the quadratic phase  $\Gamma$  and an amplitude compactly supported in an expanding ball of center the origin and radius  $O(\lambda^\delta)$ .

Combining the asymptotic expansion of the symbol  $s$  in (34), which yields

$$s(\lambda t, x', x'') \sim \lambda^d t^d \sum_{j \geq 0} \lambda^{-j} t^{-j} s_j(x', x''),$$

with the Taylor expansion of the individual factors in the amplitude of (86) in the rescaled variables, we get an asymptotic expansion of the integrand in (86) in descending powers of  $\lambda^{-1/2}$ :

$$\begin{aligned} & e^{i\sqrt{\lambda}\Gamma_x(t,u)} e^{t\mathcal{E}_x} e^{tR_3} \cdot \mathcal{D}_\lambda(y, \tau, t) \\ & \sim e^{i\sqrt{\lambda}\Gamma(t,u)} e^{t\mathcal{E}} \cdot \lambda^d \sum_{j,l \geq 0} \lambda^{-j-l/2} t^{d-j} \mathcal{P}_{j,l}(\xi, \mathbf{w}, u, \eta), \end{aligned} \quad (87)$$

where  $\mathcal{P}_{j,l}$  is a homogeneous polynomial of degree  $\leq 3l$  (dependence on  $x$  is omitted). Indeed,  $\lambda^\delta$  appears in (76) only in the rescaling of the bump function  $\gamma$ , which is identically equal to one on a neighborhood of the origin (and thus has vanishing derivatives to all orders at the origin).<sup>5</sup>

*Remark 4.1.* We have in particular  $\mathcal{P}_{0,0}(x, x) = \pi^{-d}$ , and in addition in view of (76) and the exponent  $\lambda R_3$  one concludes that  $\mathcal{P}_{j,l}$  has parity  $(-1)^\ell$ .

The remainder at step  $(j_0, l_0)$  is bounded by

$$\lambda^{-j_0-(1+l_0)/2} \mathcal{R}_{j_0,l_0}(\xi, \mathbf{w}, u, \eta) e^{-a(\|\mathbf{w}\|^2 + \|\eta\|^2 + u^2)},$$

where again  $\mathcal{R}_{j_0,l_0}$  is a polynomial of degree  $l_0$ .

Given that  $\|\xi\| = O(\lambda^\delta)$ , the previous expression is bounded above by

$$C_{j_0,l_0} \lambda^{-j_0-(1+l_0)/2+3l_0\delta} = C_{j_0,l_0} \lambda^{-j_0-1/2-3l_0(1/6-\delta)},$$

since on the other hand integration in the inner integral is over a domain of the form  $(1/(2D), 2D) \times (-c\lambda^\delta, c\lambda^\delta)$ , the expansion may be integrated term by term. Thus we get

$$I_\lambda(x, \eta, \mathbf{w}, \xi) \sim \sum_{j,l \geq 0} \lambda^{d-j-l/2} I_\lambda(x, \eta, \mathbf{w}, \xi)_{j,l}, \quad (88)$$

where

$$I_\lambda(x, \eta, \mathbf{w}, \xi)_{j,l} =: \int_{1/(2D)}^{2D} \int_{-\infty}^{+\infty} e^{i\sqrt{\lambda}\Gamma_x(t,u)} e^{t\mathcal{E}_x} \cdot t^{d-j} \mathcal{P}_{j,l}(\xi, \mathbf{w}, u, \eta) du dt. \quad (89)$$

---

<sup>5</sup>The degree of  $\mathcal{P}_{j,l}$  is bounded by  $3l$  rather than  $l$  because of the exponent  $\lambda R_3(\tau/\sqrt{\lambda}, \mathbf{w}/\sqrt{\lambda}, \xi/\sqrt{\lambda})$ .

It is immediate from (84) that  $\Gamma$  has a unique stationary point

$$P_0 = (t_0, u_0) =: \left( \frac{1}{\|\Phi(m_x)\|}, 0 \right),$$

where it vanishes; the Hessian matrix at the critical point is

$$H_{P_0}(\Gamma) = \begin{bmatrix} 0 & \|\Phi(m_x)\| \\ \|\Phi(m_x)\| & 0 \end{bmatrix},$$

with determinant and signature

$$\det(H_{P_0}(\Gamma)) = -\|\Phi(m_x)\|^2, \quad \text{sgn}(H_{P_0}(\Gamma)) = 0.$$

Therefore, the Hessian operator is given by

$$L_\Gamma =: \frac{i}{\|\Phi(m_x)\|} \frac{\partial^2}{\partial t \partial u}. \quad (90)$$

Furthermore, iterated integration by parts in  $(t, u)$  shows that only a bounded neighborhood of the critical point contributes non-negligibly to the asymptotics of  $I(\eta, \mathbf{w}, \xi)_{j,l}$ . More precisely, let  $\beta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$  be a bump function identically equal to 1 in a neighborhood of  $(t_0, u_0)$ . Then we can split (89) as

$$I(\eta, \mathbf{w}, \xi)_{j,l} = I(\eta, \mathbf{w}, \xi)'_{j,l} + I(\eta, \mathbf{w}, \xi)''_{j,l}, \quad (91)$$

where  $I(\eta, \mathbf{w}, \xi)'_{j,l}$ ,  $I(\eta, \mathbf{w}, \xi)''_{j,l}$  are given by (89), but with the integrand multiplied by  $\beta(t, u)$  and  $1 - \beta(t, u)$ , respectively. Integration by parts in  $(t, u)$  in  $I(\eta, \mathbf{w}, \xi)''_{j,l}$  as in the standard proof of the stationary phase Lemma is legitimate, because the integrand is compactly supported away from the critical point (and, at any rate, bounded by a decaying exponential in  $u$ ); on the other hand at each iteration a factor  $\lambda^{-1/2}$  is introduced, and integration is over a domain of diameter  $O(\lambda^\delta)$ , and we conclude that  $I(\eta, \mathbf{w}, \xi)''_{j,l} = O(\lambda^{-\infty})$ .

Applying the Stationary Phase Lemma to  $I(\eta, \mathbf{w}, \xi)'_{j,l}$ , we obtain an asymptotic expansion in (88) of the form

$$\begin{aligned} \lambda^{d-j-l/2} I(\eta, \mathbf{w}, \xi)_{j,l} &\sim \frac{2\pi}{\|\Phi(m_x)\|} \cdot \lambda^{d-1/2-j-l/2} \\ &\cdot \sum_{a \geq 0} \lambda^{-a/2} \frac{1}{a!} L_\Gamma^a \left( t^{d-j} e^{t\mathcal{E}_x} \cdot \mathcal{P}_{j,l}(\xi, \mathbf{w}, u, \eta) \right) \Big|_{t=t_0, u=u_0}. \end{aligned} \quad (92)$$

Given (84) and (90), we conclude that

$$L_\Gamma^a (e^{t\mathcal{E}_x}) = \mathcal{Q}_a(x, t, u; \xi, \mathbf{w}, \eta) e^{t\mathcal{E}_x}, \quad (93)$$

where  $\mathcal{Q}_a(x, t, u; \xi, \mathbf{w}, \eta)$  is a polynomial in  $(\xi, \mathbf{w}, \eta)$ , of degree  $\leq 3a$ . It follows that

$$L_\Gamma^a (t^{d-j} e^{t\mathcal{E}_x} \cdot \mathcal{P}_{j,l}(\xi, \mathbf{w}, u, \eta)) = \mathcal{R}_{j,l,a}(x, t; \xi, \mathbf{w}, u, \eta) e^{t\mathcal{E}_x}, \quad (94)$$

where  $\mathcal{R}_{j,l,a}$  is a polynomial in  $(\xi, \mathbf{w}, u, \eta)$ , of degree  $\leq 3(a+l)$ .

*Remark 4.2.* By (90), we have

$$L_\Gamma^a = \left( \frac{i}{\|\Phi(m_x)\|} \right)^a \frac{\partial^{2a}}{\partial t^a \partial u^a}.$$

Application of  $\partial^a / \partial t^a$  in (94) doesn't change the parity of the argument in  $(\xi, \mathbf{w}, u, \eta)$ , as  $\mathcal{E}_x$  is homogeneous of degree 2 (see (84)). On the other hand, for the same reason  $\partial^a / \partial u^a$  changes the parity by a factor  $(-1)^a$ . Since by Remark 4.1  $\mathcal{P}_{j,l}(\xi, \mathbf{w}, u, \eta)$  has parity  $(-1)^l$ , we conclude that  $\mathcal{R}_{j,l,a}(x, t; \xi, \mathbf{w}, u, \eta)$  has parity  $(-1)^{l+a}$ .

Returning to (86), we end up with an asymptotic expansion

$$\begin{aligned} I(\eta, \mathbf{w}, \xi) \sim & \frac{2\pi}{\|\Phi(m_x)\|} \cdot \left( \frac{\lambda}{\pi \|\Phi(m_x)\|} \right)^d \lambda^{-1/2} \\ & \cdot \exp \left( \frac{1}{\|\Phi(m_x)\|} \mathfrak{A}_x(\xi, \mathbf{w}, \eta) \right) \sum_{\ell \geq 0} \lambda^{-\ell/2} \mathcal{P}_\ell(x; \xi, \mathbf{w}, \eta), \end{aligned} \quad (95)$$

where  $\mathcal{P}_\ell$  is a polynomial of degree  $\leq 3\ell$  in  $(\xi, \mathbf{w}, \eta)$ , and parity  $(-1)^\ell$ ; in particular  $\mathcal{P}_0 = \chi(0)$ . Also,

$$\begin{aligned} \mathfrak{A}_x(\xi, \mathbf{w}, \eta) &= \mathcal{E}_x(\xi, \eta, \mathbf{w}) \\ &= \psi_2(A \mathbf{w}, \mathbf{w}) + 2i g_m(\xi_M(m_x), \eta_M(m_x)) - \frac{1}{2} \|\eta_M(m_x)\|^2. \end{aligned} \quad (96)$$

Thus,

$$\Re(\mathfrak{A}_x(\xi, \mathbf{w}, \eta)) \leq -a (\|\mathbf{w}\|^2 + \|\eta\|^2)$$

for some  $a > 0$ . On the other hand, since  $\|\xi\|, \|\eta\|, \|\mathbf{w}\| = O(\lambda^\delta)$ , we obtain on the domain of integration

$$|\lambda^{-\ell/2} \mathcal{P}_\ell(x; \xi, \mathbf{w}, \eta)| \leq C_\ell \lambda^{-\ell/2+3\delta\ell} = C_\ell \lambda^{-3\ell(1/6-\delta)},$$

and a similar bound for the remainder; since the domain of integration is again a ball centered at the origin of radius  $O(\lambda^\delta)$ , that the expansion can again be integrated term by term in  $d\eta$ .

We conclude that (85) may be rewritten as an asymptotic expansion

$$\begin{aligned} \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y_\lambda, y_\lambda) &\sim \frac{2\pi}{\|\Phi(m_x)\|} \cdot \left( \frac{\lambda}{\pi \|\Phi(m_x)\|} \right)^d \lambda^{\frac{1-r}{2}} e^{-i\lambda \langle \beta, \mathbf{s}_0 \rangle} \\ &\cdot e^{\psi_2(A\mathbf{w}, \mathbf{w})/\|\Phi(m_x)\|} \cdot \sum_{\ell \geq 0} \lambda^{-\ell/2} I_\ell(x; \mathbf{w}, \xi), \end{aligned} \quad (97)$$

where for  $\ell = 0, 1, \dots$  we have set

$$\begin{aligned} I_\ell(x; \mathbf{w}, \xi) & \\ =: \int_{\ker \Phi(m_x)} \mathcal{P}_\ell(x; \xi, \mathbf{w}, \eta) e^{\frac{1}{\|\Phi(m_x)\|} \left[ 2i g_m(\xi_M(m_x), \eta_M(m_x)) - \frac{1}{2} \|\eta_M(m_x)\|^2 \right]} d\eta; \end{aligned} \quad (98)$$

here  $d\eta$  is the Lebesgue measure on  $\ker \Phi(m_x) \subseteq \mathfrak{t}$ , when the latter subspace is identified with  $\mathbb{R}^{r-1}$  by means of an orthonormal basis.

To compute the latter Gaussian integral, let us choose orthonormal basis  $\mathcal{K}$  for  $\ker \Phi(m_x)$  and  $\mathcal{D}$  for the subspace  $V(m_x) =: \text{val}_{m_x}(\ker \Phi(m_x)) \subseteq T_{m_x}M$ , and let  $C$  be the  $(r-1) \times (r-1)$  invertible matrix representing the isomorphism  $\ker \Phi(m_x) \rightarrow V(m_x)$  induced by  $\text{val}_{m_x}$  with respect to these basis. If  $\mathbf{u}_\xi, \mathbf{u}_\eta \in \mathbb{R}^{r-1}$  are the coordinate vectors of  $\xi, \eta \in \ker \Phi(m_x)$  with respect to  $\mathcal{K}$ , we have

$$g_m(\xi_M(m_x), \eta_M(m_x)) = \mathbf{u}_\xi^t C^t C \mathbf{u}_\eta,$$

so that the matrix  $D$  and the function  $\mathcal{D}$  in Definition 1.8 are given by  $D = C^t C$  and  $\mathcal{D}(m_x) = |\det(C)|$ , respectively.

On the other hand, the basis  $\mathcal{K}$  provides a unitary isomorphism  $\mathbb{R}^{r-1} \cong \ker \Phi(m_x)$ , and we can convert the integral in  $d\eta$  over  $\ker \Phi(m_x)$  into an integral in  $d\mathbf{u}$  over  $\mathbb{R}^{r-1}$ :

$$\int_{\ker \Phi(m_x)} d\eta \rightarrow \int_{\mathbb{R}^{r-1}} d\mathbf{u}.$$

With the change of variables  $\mathbf{a} = \mathbf{a}(\mathbf{u}) =: C\mathbf{u}/\sqrt{\|\Phi(m_x)\|}$ , we can rewrite (98) as follows:

$$\begin{aligned} I_\ell(x, \mathbf{w}, \xi) & \\ = \int_{\mathbb{R}^{r-1}} \mathcal{P}_\ell(x; \xi, \mathbf{w}, \mathbf{u}) \exp \left( \frac{1}{\|\Phi(m_x)\|} \left[ 2i \langle C \mathbf{u}_\xi, C \mathbf{u} \rangle - \frac{1}{2} \|C \mathbf{u}\|^2 \right] \right) d\mathbf{u} & \\ = \frac{\|\Phi(m_x)\|^{(r-1)/2}}{|\det(C)|} \int_{\mathbb{R}^{r-1}} \mathcal{Q}_\ell(x; \xi, \mathbf{w}, \mathbf{a}) \exp \left( \left[ i \left\langle \frac{2 C \mathbf{u}_\xi}{\sqrt{\|\Phi(m_x)\|}}, \mathbf{a} \right\rangle - \frac{1}{2} \|\mathbf{a}\|^2 \right] \right) d\mathbf{a}; \end{aligned} \quad (99)$$

here  $\mathcal{Q}_\ell(x; \cdot, \cdot, \cdot)$  is obtained from  $\mathcal{P}_\ell(x; \cdot, \cdot, \cdot)$  by the change of variable  $\mathbf{u} = \mathbf{u}(\mathbf{a})$ , and is therefore a polynomial of degree  $\leq 3\ell$ , and parity  $(-1)^\ell$ .

Now the latter integral may be interpreted as the application of a differential polynomial  $\tilde{\mathcal{Q}}_\ell(x; \xi, \mathbf{w}, D_\xi)$  in  $D_\xi = -i\partial_\xi$  of collective degree  $\leq 3\ell$  in  $(\xi, \mathbf{w}, D_\xi)$  to the exponential  $\exp(-\|\mathbf{a}\|^2/2)$ , evaluated at  $2C\mathbf{u}_\xi/\sqrt{\|\Phi(m_x)\|}$ . More explicitly,

$$\begin{aligned} I_\ell(x, \mathbf{w}, \xi) &= \frac{1}{\mathcal{D}(m_x)} (2\pi \|\Phi(m_x)\|)^{(r-1)/2} \tilde{\mathcal{Q}}_\ell(x; \xi, \mathbf{w}, D_\xi) \left( \exp \left( -\frac{2\|C\mathbf{u}_\xi\|^2}{\|\Phi(m)\|} \right) \right) \\ &= \frac{1}{\mathcal{D}(m_x)} (2\pi \|\Phi(m_x)\|)^{(r-1)/2} \mathcal{R}_\ell(x; \xi, \mathbf{w}) \cdot \exp \left( -\frac{2\|\xi_M(m_x)\|^2}{\|\Phi(m)\|} \right); \end{aligned} \quad (100)$$

here again  $\mathcal{R}_\ell(x; \cdot, \cdot)$  is a polynomial of degree  $\leq 3\ell$  and degree  $(-1)^\ell$ , and  $\mathcal{R}_0 = \chi(0)$ . The norm of  $\xi_M(m_x)$  in the latter line is of course computed in  $T_{m_x}M$ .

Inserting (100) in (97) we end up with the asymptotic expansion

$$\begin{aligned} \mathcal{S}_\chi(\lambda\beta, \mathbf{s}_0, y_\lambda, y_\lambda) & \sim \frac{2^{\frac{r+1}{2}} \pi}{\|\Phi(m_x)\|} \cdot \left( \frac{\lambda}{\pi \|\Phi(m_x)\|} \right)^{d+\frac{1-r}{2}} \frac{e^{-i\lambda \langle \beta, \mathbf{s}_0 \rangle}}{\mathcal{D}(m_x)} e^{[\psi_2(A\mathbf{w}, \mathbf{w}) - 2\|\xi_M(m_x)\|^2]/\|\Phi(m_x)\|} \\ & \cdot \sum_{\ell \geq 0} \lambda^{-\ell/2} \mathcal{R}_\ell(x; \xi, \mathbf{w}). \end{aligned} \quad (101)$$

Since (101) coincides with (25) with  $\mathbf{n} = J_{m_x}(\xi_M(m_x))$ , this completes the proof of Theorem 1.2.  $\square$

*Proof of Corollary 1.2.* To ease the exposition, let us pretend that  $M_\beta(\mathbf{s}_0)$  is connected; otherwise we merely need to repeat the argument over each connected component.

Let us write as above  $y$  in the neighborhood of  $X_\beta(\mathbf{s}_0)$  as  $y = x + \mathbf{v}$ , where  $x \in X_\beta(\mathbf{s}_0)$  and  $\mathbf{v} \in N_x(X_\beta(\mathbf{s}_0))$  is in (23). Thus we are assuming a moving system of HLC, which is in general only possible locally along  $X_\beta(\mathbf{s}_0)$ . So to make the argument complete we should introduce an open cover of  $X_\beta(\mathbf{s}_0)$  and a partition of unity subordinate to it, but we shall leave this implicit to ease the exposition. By the given choice of HLC, we can unitarily identify

$$N_x(X_\beta(\mathbf{s}_0)) \cong N_{m_x}(M(\mathbf{s}_0)) \oplus N_{m_x}(M_\beta) \cong \mathbb{C}^c \oplus \mathbb{R}^{r-1},$$

where  $c$  is the complex codimension of  $M(\mathbf{s}_0)$  in  $M$ .

We have, by (19) and Theorem 1.1,

$$\begin{aligned}\mathcal{F}(\chi_{\mathbf{s}_0} \cdot \text{tr}(\mathfrak{U}))(\lambda \beta) &= \int_X \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, y, y) \, dV_X(y) \\ &\sim \int_{X_\beta(\mathbf{s}_0)} F_{\mathbf{s}_0}(\lambda \beta, x) \, dV_{X_\beta(\mathbf{s}_0)}(x),\end{aligned}\quad (102)$$

where

$$\begin{aligned}F_{\mathbf{s}_0}(\lambda \beta, x) &=: \int_{\mathbb{R}^{r-1}} \int_{\mathbb{C}^c} \mathcal{S}_\chi(\lambda \beta, \mathbf{s}_0, x + (\mathbf{w} + \mathbf{n}), x + (\mathbf{w} + \mathbf{n})) \\ &\quad \cdot \varrho'(\lambda^{1/2-\delta} \mathbf{w}) \, \varrho''(\lambda^{1/2-\delta} \mathbf{n}) \, d\mathbf{w} \, d\mathbf{n};\end{aligned}\quad (103)$$

here  $\varrho' \in \mathcal{C}_0^\infty(\mathbb{C}^c)$  and  $\varrho'' \in \mathcal{C}_0^\infty(\mathbb{R}^{r-1})$  are bump functions identically equal to 1 on a neighborhood of the origin. In turn, applying the rescaling  $\mathbf{w} \mapsto \mathbf{w}/\sqrt{\lambda}$  and  $\mathbf{n} \mapsto \mathbf{n}/\sqrt{\lambda}$ , we can rewrite (103) as follows:

$$F_{\mathbf{s}_0}(\lambda \beta, x) = \lambda^{-c+\frac{1-r}{2}} \mathcal{F}_{\mathbf{s}_0}(\lambda \beta, x), \quad (104)$$

where

$$\begin{aligned}\mathcal{F}_{\mathbf{s}_0}(\lambda \beta, x) &=: \int_{\mathbb{R}^{r-1}} \int_{\mathbb{C}^c} \mathcal{S}_\chi\left(\lambda \beta, \mathbf{s}_0, x + \left(\frac{\mathbf{w}}{\sqrt{\lambda}} + \frac{\mathbf{n}}{\lambda}\right), x + \left(\frac{\mathbf{w}}{\sqrt{\lambda}} + \frac{\mathbf{n}}{\lambda}\right)\right) \\ &\quad \cdot \varrho'(\lambda^{-\delta} \mathbf{w}) \, \varrho''(\lambda^{-\delta} \mathbf{n}) \, d\mathbf{w} \, d\mathbf{n};\end{aligned}$$

Here integration is over a ball centered at the origin and radius  $O(\lambda^\delta)$ , and the integrand is given by (101) with  $\mathbf{n}$  in place of  $J_m(\xi_M(m_x))$ . It thus follows that  $\mathcal{F}_{\mathbf{s}_0}(\lambda \beta, x)$  is given by an asymptotic expansion in descending powers of  $\lambda^{1/2}$ . In addition, since  $\mathcal{R}_\ell$  has parity  $(-1)^\ell$ , only even  $\ell$ 's give a non-vanishing contribution; therefore, the resulting integrated asymptotic expansion is really in descending powers of  $\lambda$ .

More explicitly, we get

$$\begin{aligned}\mathcal{F}_{\mathbf{s}_0}(\lambda \beta, x) &\sim \frac{2^{\frac{r+1}{2}} \pi}{\|\Phi(m_x)\|} \cdot \left( \frac{\lambda}{\pi \|\Phi(m_x)\|} \right)^{d+\frac{1-r}{2}} \frac{e^{-i\lambda \langle \beta, \mathbf{s}_0 \rangle}}{\mathcal{D}(m_x)} \\ &\quad \cdot \sum_{k \geq 0} \lambda^{-k} \mathcal{L}_{\mathbf{s}_0, k}(\lambda \beta, x),\end{aligned}\quad (105)$$

where

$$\mathcal{L}_{\mathbf{s}_0, k}(\lambda \beta, x) =: \int_{\mathbb{R}^{r-1}} \int_{\mathbb{C}^c} \mathcal{S}_k(x; \mathbf{n}, \mathbf{w}) e^{[\psi_2(A\mathbf{w}, \mathbf{w}) - 2\|\mathbf{n}\|^2]/\|\Phi(m_x)\|} \, d\mathbf{w} \, d\mathbf{n};$$

here  $\mathcal{S}_k(x; \mathbf{n}, \mathbf{w}) =: \mathcal{R}_{2k}(x; \xi, \mathbf{w})$  with  $\mathbf{n} = J_{m_x}(\xi_M(m_x))$ , so  $\mathcal{S}_0(x; \mathbf{n}, \mathbf{w}) = \chi(\mathbf{0})$ .

Let us compute the leading order term in (105). To this end, let  $A'$  be the unitary  $c \times c$  matrix representing the restriction of  $d_{m_x} \phi_{-\mathbf{s}_0}^M$  to the normal bundle  $N_{m_x}(M(\mathbf{s}_0)) \cong \mathbb{C}^c$  with respect to a *complex* orthonormal basis. We have

$$\begin{aligned}
\mathcal{L}_{\mathbf{s}_0,0} &= \chi(\mathbf{0}) \cdot \int_{\mathbb{R}^{r-1}} \int_{\mathbb{C}^d} e^{[\psi_2(A\mathbf{w}, \mathbf{w}) - 2\|\xi_M(m_x)\|^2]/\|\Phi(m_x)\|} d\mathbf{w} d\mathbf{n} \\
&= \chi(\mathbf{0}) \cdot \left( \int_{\mathbb{C}^d} e^{\psi_2(A\mathbf{w}, \mathbf{w})/\|\Phi(m_x)\|} d\mathbf{w} \right) \cdot \left( \int_{\mathbb{R}^{r-1}} e^{-2\|\mathbf{n}\|^2/\|\Phi(m_x)\|} d\mathbf{n} \right) \\
&= \chi(\mathbf{0}) \cdot \|\Phi(m_x)\|^{\frac{r-1}{2}+c} \left( \int_{\mathbb{C}^c} e^{\psi_2(A'\mathbf{u}, \mathbf{u})} d\mathbf{u} \right) \cdot \left( \int_{\mathbb{R}^{r-1}} e^{-2\|\mathbf{a}\|^2} d\mathbf{a} \right) \\
&= \chi(\mathbf{0}) \cdot \|\Phi(m_x)\|^{\frac{r-1}{2}+c} \frac{\pi^c}{\det(I_c - A')} \cdot \left( \frac{\pi}{2} \right)^{\frac{r-1}{2}} \quad (106)
\end{aligned}$$

(see (64) of [P1]). Inserting (106) in (105), we obtain

$$\begin{aligned}
F_{\mathbf{s}_0}(\lambda \beta, x) &\sim \lambda^{-c+\frac{1-r}{2}} \frac{2^{\frac{r+1}{2}} \pi}{\|\Phi(m_x)\|} \cdot \left( \frac{\lambda}{\pi \|\Phi(m_x)\|} \right)^{d+\frac{1-r}{2}} \frac{e^{-i\lambda \langle \beta, \mathbf{s}_0 \rangle}}{\mathcal{D}(m_x)} \\
&\cdot \chi(\mathbf{0}) \cdot \|\Phi(m_x)\|^{\frac{r-1}{2}+c} \frac{\pi^c}{\det(I_c - A')} \cdot \left( \frac{2}{\pi} \right)^{\frac{1-r}{2}} \\
&\cdot \sum_{k \geq 0} \lambda^{-k} \mathcal{U}_k(\beta, x) \\
&= \frac{2\pi}{\|\Phi(m_x)\|} \frac{e^{-i\lambda \langle \beta, \mathbf{s}_0 \rangle}}{\mathcal{D}(m_x)} \left( \frac{\lambda}{\|\Phi(m_x)\| \pi} \right)^{d+1-r-c} \cdot \frac{1}{\det(I_c - A')} \\
&\cdot \sum_{k \geq 0} \lambda^{-k} \mathcal{U}_k(\mathbf{s}_0, \beta, x) \quad (107)
\end{aligned}$$

where  $\mathcal{U}_0(\beta, x) = \chi(\mathbf{0})$ . Clearly  $\det(I_c - A') = \mathbf{c}(\mathbf{s}_0)$ . Therefore, using (107) in (102) we obtain

$$\mathcal{F}(\chi_{\mathbf{s}_0} \cdot \text{tr}(\mathfrak{U}))(\lambda \beta) \sim \frac{2\pi}{\mathbf{c}(\mathbf{s}_0)} e^{-i\lambda \langle \beta, \mathbf{s}_0 \rangle} \left( \frac{\lambda}{\pi} \right)^{d+1-r-c} \cdot \sum_{k \geq 0} \lambda^{-k} \mathcal{U}_k(\mathbf{s}_0, \beta),$$

with the leading order coefficient being given by

$$\mathcal{U}_k(\mathbf{s}_0, \beta) =: \chi(\mathbf{0}) \cdot \int_{X_\beta(\mathbf{s}_0)} \frac{1}{\|\Phi(m_x)\|^{d+2-r-c}} \frac{1}{\mathcal{D}(m_x)} dV_{X_\beta(\mathbf{s}_0)}(x).$$

□



## 5 Notational Appendix

For the reader's convenience, we collect here some of the notation going into the arguments and asymptotic expansions.

1.  $(M, J, 2\omega)$ : the Hodge manifold playing the role of 'phase space'.
2.  $v_f$ : the Hamiltonian vector field associated to  $f \in \mathcal{C}^\infty(M)$ .
3.  $\phi_S^M : M \rightarrow M$  ( $S \in \mathbb{R}$ ): the Hamiltonian flow of  $v_f$  (dependence on  $f$  is understood).
4.  $(\mathcal{A}, h)$ : the positive Hermitian holomorphic line bundle quantizing  $(M, 2\omega)$ , with dual  $\mathcal{A}^\vee$ .
5.  $X \subseteq \mathcal{A}^\vee$ : the unit circle bundle.
6.  $\alpha$ : the contact form on  $X$ .
7.  $dV_M$  and  $dV_X$ : the naturally induced volume forms on  $M$  and  $X$ , respectively.
8.  $\mathbf{v}^\sharp$ : the horizontal lift (for  $\alpha$ ) of a tangent vector to  $M$ ;  $\partial_\theta$ : the generator of the circle action on  $X$  (see (2) and (32)).
9.  $\tilde{v}_f =: v_f^\sharp - f \partial_\theta$ : the contact vector field on  $X$  associated to  $f \in \mathcal{C}^\infty(M)$  (see (2)).
10.  $\phi_s^X : X \rightarrow X$  ( $s \in \mathbb{R}$ ): the contact flow generated by  $\tilde{v}_f$ .
11.  $H(X) \subseteq L^2(X)$ : the Hardy space of  $X$  (Definition 1.3).
12.  $\Pi : L^2(X) \rightarrow L^2(X)$ : the Szegő kernel of  $X$  (Definition 1.3).
13.  $t\psi(x, y)$  and  $s(t, x, y)$ : the phase and amplitude in the description of  $\Pi$  as an FIO after [BtSj] (see (34)).
14.  $\mathfrak{U}(s) = \mathfrak{U}_f(s) : H(X) \rightarrow H(X)$  ( $s \in \mathbb{R}$ ): the 1-parameter family of unitary automorphisms induced by a compatible  $f \in \mathcal{C}^\infty(M)$  (see (3)).
15.  $\mathfrak{T}_f =: i\tilde{v}_f|_{H(X)} : H(X) \rightarrow H(X)$ : the self-adjoint Toeplitz operator associated to the contact vector field of a compatible Hamiltonian  $f$ , with principal symbol  $\mathfrak{s}_{\mathfrak{T}_f}(x, r \alpha_x) = r f(\pi(x))$  (see (4)).
16.  $\text{tr}(\mathfrak{U})$ : the distributional trace of  $\mathfrak{U}$  (§1.1.2).

17.  $v_k$  and  $\tilde{v}_k$ : the commuting Hamiltonian and contact vector fields associated to Poisson commuting compatible Hamiltonians  $f_k$ ,  $k = 1, \dots, r$  (§1.1.3).
18.  $\phi_s^M : M \rightarrow M$  and  $\phi_s^X : X \rightarrow X$  ( $s \in \mathbb{R}^r$ ): the Hamiltonian and contact actions of  $\mathbb{R}^r$  on  $M$  and  $X$ , respectively, generated by the  $f_k$ 's.
19.  $\mathfrak{T}_k$ : the Toeplitz operator induced by restriction of  $i\tilde{v}_k$ ;  $\mathfrak{T} =: (\mathfrak{T}_k)$ , the corresponding commuting system of Toeplitz operators.
20.  $\Lambda_j = (\lambda_{kj})$ : the  $j$ -th joint eigenvalue of  $\mathfrak{T} = (\mathfrak{T}_k)$ , with joint eigenfunction  $e_j$  (see §1.1.4).
21.  $\mathfrak{t} =: T_0\mathbb{R}^r$ ,  $\mathfrak{t}^\vee$  its dual (see Notation 1.1).
22.  $\xi_M$  and  $\xi_X$ : the vector fields on  $M$  and  $X$ , respectively, induced by  $\xi \in \mathfrak{t}$  (Definition 1.6).
23.  $\text{val}_m : \mathfrak{t} \rightarrow T_m M$  and  $\text{val}_x : \mathfrak{t} \rightarrow T_x X$ : the evaluation maps  $\xi \mapsto \xi_M(m)$  and  $\xi \mapsto \xi_X(x)$ , respectively (Definition 1.6).
24.  $\Phi : M \rightarrow \mathfrak{t}^\vee$ : the moment map associated to the Hamiltonian action of  $\mathbb{R}^r$  generated by the  $f_j$ 's (see (7) and Notation 1.1 in §1.1.5).
25.  $\Xi : M \rightarrow \mathfrak{t}$ : the normalized 'dual map' to  $\Phi$  (§2.1.2).
26.  $\mathfrak{U}(s) : H(X) \rightarrow H(X)$  ( $s \in \mathbb{R}^r$ ): the unitary representation of  $\mathbb{R}^r$  associated to the compatible and commuting  $f_j$ 's (see (8)).
27.  $\text{tr}(\mathfrak{U})$ : the distributional trace of  $\mathfrak{U}$  (see (12)).
28.  $\text{Per}(\phi^M)$  and  $\text{Per}(\phi^X)$ : the set of periods of  $\phi^M$  and  $\phi^X$ , respectively (see (13) and Definition 1.10).
29.  $M(s)$  and  $X(s)$ : the fixed loci of  $\phi_s^M$  and  $\phi_s^X$ , respectively (Definition 1.9).
30.  $\beta \in (\mathbb{R}^r)^\vee$ : a general covector of unit norm at the origin of  $\mathbb{R}^r$  (see (14) and Definition 1.8).
31.  $M_\beta =: \pi^{-1}(\mathbb{R}_+ \cdot \beta)$ ,  $X_\beta =: (\Phi \circ \pi)^{-1}(\mathbb{R}_+ \cdot \beta)$  (see Definition 1.7);  $N(M_\beta)$  and  $N(X_\beta)$ : their normal bundles (see (22), (32), and §2.1.5).
32.  $M_\beta(s) =: M_\beta \cap M(s)$ ,  $X_\beta(s) =: X_\beta \cap M(s)$  (Definition 1.10).
33.  $N(X_\beta(s_0))$ : the normal bundle of  $X_\beta(s_0)$  (see (22)).

- 34.  $dV_{M_\beta(\mathbf{s}_0)_j}$ : the Riemannian volume density on the  $j$ -th connected component  $M_\beta(\mathbf{s}_0)_j$  of  $M_\beta(\mathbf{s}_0)$  (Corollary 1.2).
- 35.  $f_j$ : the complex dimension of  $M(\mathbf{s}_0)$  along  $M_\beta(\mathbf{s}_0)_j$  (Definition 1.12).
- 36.  $\mathfrak{c}_j(\mathbf{s}_0)$ : the Poincaré type invariant along  $M_\beta(\mathbf{s}_0)_j$  (Definition 1.12).
- 37.  $\mathcal{F}$ : the Fourier transform on  $\mathbb{R}^r$  (see (14)).
- 38.  $\chi$ : a bump function on  $\mathbb{R}^r$  supported near the origin;  $\chi_{\mathbf{s}_0}(\cdot) =: \chi(\cdot - \mathbf{s}_0)$  its translate (see (14));  $\widehat{\chi}$  its Fourier transform.
- 39.  $\mathcal{S}_\chi(\lambda\beta, \mathbf{s}_0)$ : the smoothing operator obtained by averaging  $\mathfrak{U}(\mathbf{s})$  with weight  $\chi_{\mathbf{s}_0}(\mathbf{s}) e^{-i\lambda\langle\beta, \mathbf{s}\rangle}$  (see (17)).
- 40.  $\mathcal{D}(m)$ : the invariant relating the two naturally induced Euclidean structures on  $\ker \Phi(m)$  when  $m \in M_\beta$  (Definition 1.8).
- 41.  $x+(\theta, \mathbf{v})$ : the additive notation for Heisenberg local coordinates (§1.1.7).
- 42.  $\psi_2(\mathbf{v}, \mathbf{w})$ : the universal exponent from [SZ] governing Szegő kernel scaling asymptotics (Definition 1.11).
- 43.  $A = A_{m_x}$ : the unitary matrix representing  $d_x\phi_{-\mathbf{s}_0}^X : T_{m_x}M \rightarrow T_{m_x}M$ , given a choice of a HLC system centered at  $x \in X_\beta(\mathbf{s}_0)$  (Notation 1.2 in §1.1.7).

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